PhD. Dissertation

GALOISIAN APPROACH TO SUPERSYMMETRIC QUANTUM MECHANICS

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We certify that this dissertation has been
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with our supervision and co-direction.

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Juan J. Morales-Ruiz Jacques-Arthur Weil

Todo lo puedo en Cristo
que me fortalece.

Filipenses 4, 13.

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Contents

Introduction

Structure of the Thesis

The main object studied in this thesis, in the differential Galois theory framework, is the one-dimensional stationary non-relativistic Schrödinger equation

$$\partial_x^2\Psi = (V-\lambda)\Psi, \quad V\in K,$$

where K is a differential field containing x, closed algebraically and of characteristic zero. In particular, K is considered as the smallest differential field containing the potential V.

This thesis is divided in two parts.

Chapter 1. Theoretical Background

In this part there are not original results. Summaries of Picard-Vessiot theory and supersymmetric quantum mechanics, necessaries to understand the next chapter, are presented here.

Chapter 2. Differential Galois Theory Approach to Supersymmetric Quantum Mechanics

This part contain the original results of this thesis, which were developed using the previous chapter. Up to specific cases; theorems, propositions, corollaries and lemmas given in this chapter are considered as original results of this thesis.

Two different Galoisian approaches are studied in this chapter, which depends on the differential field: the first one is $\mathbb{C}(x)$ and the second one is $K = \mathbb{C}(z(x), \partial_z(x))$, where $z = z(x)$ is a *Hamiltonian change of variable.* This concept allow us to introduce an useful derivation $\widehat{\partial}_z$, important tool to transforms differential equations with non rational coefficients into differential equations with rational coefficients, to apply the results given in the case of $\mathbb{C}(x)$.

This chapter is divided in three parts or sections.

Section 2.1

Here is introduced the set Λ as the set of values of λ in which the Schrödinger equation is integrable in the sense of Picard-Vessiot theory. With this new set Λ, we define in an easy way the concepts of *algebraically solvable*, *algebraically quasi-solvable* and *algebraically non-solvable* potentials.

Also we define in this section the concepts of *iso-galoisian*, *virtually iso-galoisian* and *strong iso-galoisian* transformations. So, we give in proposition 2.1.5 some conditions in which the transformation of the second order linear differential equation into a reduced form (the term in $\partial_x y$ is absent) is strong iso-galoisian or virtually strong iso-galoisian. In particular case, corollary 2.1.7 shows that the reduction of Sturm-Liouville problem is a virtually strong iso-galoisian transformation.

Section 2.2

This section is devoted to the analysis of supersymmetric quantum mechanics from a differential Galois theory point of view when V is a rational function. We start considering the polynomial case proving that the only one possibility to get algebraically solvable potential is when V is a polynomial of degree 2. In case of algebraically quasi-solvable potentials are presented as examples the well known quartic and sextic anharmonic oscillators. Some results such as lemma 2.2.1 and theorem 2.2.2 were previously published in [5] and used in [4].

Kovacic's algorithm is applied to solve the some Schrödinger equations with rational potential (shape invariant potentials) obtaining also their differential differential Galois groups and eigenrings. In proposition 2.2.6 and corollary 2.2.7 is shown that the Schrödinger equation with rational potential does not fall in case 3 of Kovacic's algorithm.

Darboux transformation and Crum iteration are written in a Galoisian sense (theorem 2.2.8, proposition 2.2.11 and proposition 2.2.12), so that by proposition 2.2.9 and proposition 2.2.10, such transformations can be seen as isogaloisian and for instance they preserves the Eigenrings. We prove in proposition 2.2.13 that the supersymmetric partner potentials and the superpotential are rational functions and in particular, corollary 2.2.14 says that the Darboux transformation is strong iso-galoisian when the superpotential is a rational function.

Finally, we define, in the context of differential Galois theory, the concept of shape invariant potentials giving an algorithm to check whether a potential

satisfy the shape invariance condition (remark 2.2.16) and providing an important result (theorem 2.2.17) in where the Schrödinger equations obtained with the procedure of shape invariant potentials preserves the differential Galois groups and the Eigenrings.

Section 2.3

This section is devoted to the supersymmetric quantum mechanics with non-rational potentials. We start considering the proper pullback between two second order linear differential equations in which one of them has rational coefficients (*algebrization process* when the first differential equation with non-rational coefficients is transformed into a differential equation with rational coefficients). Proposition 2.3.2 shows that the change of independent variable preserves the connected identity component of the differential Galois group and give one formula of the transformed equation (see also [5]).

In proposition 2.3.3, are presented the relationships between the differential fields, Picard-Vessiot extensions and differential Galois groups corresponding to this change of independent variable. Using these propositions, in remark 2.3.4 we introduce the concept of *Hard algebrization* giving an algorithm to obtain it whether is possible. Proposition 2.3.5, which also appears in [5], shows the algebrization of some families of second order differential equations in where the differential field is given by $\mathbb{C}(x, e^{\int f})$, being $f \in \mathbb{C}(x)$.

The definition of *Hamiltonian change of variable* is introduced, leading to the concept of *Hamiltonian algebrization*, which allows to arrive to one systematic algebrization of second order linear differential equations presented in proposition 2.3.9, remark 2.3.10 and proposition 2.3.11 (these results also can be found in [5] and were applied in [2, 3]). We show through examples that the algebrization process can be used as transformation between differential equations without expecting to obtain differential equations with rational coefficients.

We introduce a new derivation $\widehat{\partial}_z = \sqrt{\alpha}\partial_z$, in where $z = z(x)$ is a Hamiltonian change of variable and $\partial_x z = \sqrt{\alpha}$. The most important theoretical result here is that the differential field $K = \mathbb{C}(z(x), \partial_x z(x))$ is isomorphic to the differential field $\widehat{K} = \mathbb{C}(z, \sqrt{\alpha}) \supset \mathbb{C}(z)$, which allows to preserve the differential Galois groups and Eigenrings in the algebrization process (theorem 2.3.13 and proposition 2.3.16). So, we transform a lot of differential equations with non-rational coefficients into differential equations with rational coefficients (Riccati, systems, etc...) and in the case of second order linear differential equations, Kovacic's algorithm can be used successfully.

We recover the results of section 2.2 rewriting everything with $\widehat{\ }$ (hat over the symbols used in section 2.2), in this way the *algebrized supersymmetric quan-*

tum mechanics is presented with its elements: *algebrized Schrödinger operator* $\widehat{H}$, *algebrized superpotential* $\widehat{W}$, *algebrized supersymmetric partner potentials* $\widehat{V}_{\pm}$, *algebrized shape invariant potentials*, *algebrized Darboux transformation* $\widehat{\mathrm{DT}}$, *algebrized Crum iteration* $\widehat{\mathrm{CI}}_n$, *algebrized ladder operators* $\widehat{A}, \widehat{A}^{\dagger}$, *algebrized wave functions* $\widehat{\Psi}$, etc.. An important fact is given in theorem 2.3.20, i.e., $\mathrm{DT}\varphi = \varphi\widehat{\mathrm{DT}}$, where φ is the Hamiltonian algebrization.

Using the Hamiltonian algebrization and Kovacic's algorithm we solve some Schrödinger equations with non-rational potentials (shape invariant potentials), obtaining the differential Galois groups and the Eigenrings of these differential equations.

Finally, we give a mechanism to search new exactly solvable potentials through the inverse process of the Hamiltonian algebrization, using known parameterized differential equations.

Historical Outline

This historical outline begins with two mathematicians: Gastón Darboux and Emile Picard. Darboux published in 1882 the paper [28] in where he presents a proposition in a general way, which in particular case the history proved to be a notable theorem today known as *Darboux transformation.* Darboux had shown that whenever one knows to integrate the equation

$$\partial_x^2 y = (f(x) + m)y$$

for all the values of the constant m, one can obtain an infinite set of equations, displaying the variable parameter in the same way, which are integrable for any value of the parameter. This proposition also can be found in his book [29, p. 210].

One year after, in 1883, Picard published the paper [73] in which he gave the starting point to a *Galois theory for linear differential equations.* Although the analogies between the linear differential equations and the algebraic equations for a long time were announced and continued in different directions, Picard developed an analogue theory to the Galois theory for algebraic equations, arriving to a proposition which seems to correspond to the fundamental Galois theorem, in where he introduces the concept of *group of linear transformations corresponding to the linear differential equation*, which today is known as *Differential Galois Group* (the group of differential automorphism leaving fixed the elements of the field base). Another contribution of Picard to this Galois theory was the paper [72] in 1887.

Five years after, in 1892, Ernest Vessiot, doctoral former student of Picard, published his thesis [103] giving consolidation to the new Galois theory for linear

differential equations, the so-called *Picard-Vessiot theory*. Two years later, in 1894, Picard published the paper [74], summarizing the results presented in [72, 73, 103] which also can be found in his book [75, §7].

Curiously, Picard-Vessiot theory and Darboux transformation were forgotten during decades. The Picard-Vessiot theory was recovered by Joseph Fels Ritt (in 1950, see [85]), Irving Kaplansky (in 1957, see [53]), and fundamentally by Ellis Kolchin (in 1948, see [55] and references therein). Kolchin wrote the *Differential Galois Theory* in a modern language (algebraic group theory).

Darboux transformation was presented as an exercise in 1926 by Ince (see exercises 5, 6 and 7 [49, p. 132]), which follows closely the formulation of Darboux given in [28, 29].

In 1930, P. Dirac publishes *The Principles of Quantum Mechanics*, in where he gave a mathematically rigorous formulation of quantum mechanics.

In 1938, J. Delsarte wrote the paper [30], in which he introduced the notion of transformation (transmutation) operator, today know as *intertwining operator* which is closely related with Darboux transformation and ladder operators.

In 1941, E. Schrödinger published the paper [89] in which he factorized in several ways the hypergeometric equation. This was a byproduct of his *factorization method* originating an approach that can be traced back to Dirac's raising and lowering operators for the harmonic oscillator.

Ten years later, in 1951, another factorization method was presented. L. Infeld and T. E. Hull published the paper [50] in where they gave the classification of their factorizations of linear second order differential equations for eigenvalue problems of wave mechanics.

In 1955, M.M. Crum inspired in the Liouville's work about Sturm-Liouville systems (see [61, 62]), published the paper [27] giving one kind of iterative generalization of Darboux transformation. Crum surprisingly did not mention Darboux.

In 1971, G.A. Natanzon published the paper [69], in which he studied a general form of the transformation that converts the hypergeometric equation to the Schrödinger equation writing down the most general *solvable potential*, potential for which the Schrödinger equation can be reduced to hypergeometric or confluent hypergeometric form, a concept introduced by himself.

Almost one hundred years later than Darboux's proposition, in 1981, Edward Witten in his renowned paper [107] gave birth to the *Supersymmetric Quantum Mechanics*, discussing general conditions for dynamical supersymmetry breaking.

Since the work of Witten, thousands of papers, about supersymmetric quantum mechanics, has been written. We mention here some relevant papers.

In 1983, L. É. Gendenshtein published the paper [37] in where the *Shape invariance* condition, i.e. preserving the shape under Darboux transformation, was presented and used to find the complete spectra for a broad class of problems including all known exactly solvable problems of quantum mechanics (bound state and reflectionless potentials). Today this kind of exactly solvable potentials satisfying the shape invariance condition are called *Shape invariant potentials*.

In 1986, A. Turbiner in [102] introduces the concept of *quasi-exactly solvable potentials*, giving an example that is well known as *Turbiner's potential.*

In 1991, V.B. Matveev and M. Salle published the book [66] in where they focused on Darboux transformations and their relation with solitons. Matveev and Salle interpreted the Darboux transformation as Darboux covariance of a Sturm-Liouville problem and also proved that Witten's supersymetric quantum mechanics is equivalent to a single Darboux transformation.

In 1996, C. Bender and G. Dunne studied the *sextic anharmonic oscillator* in [11], which is a quasi-exactly solvable model derived from the Turbiner's potentials. They found that a portion of the spectrum correspond to the roots of polynomials in the energy. These polynomials are orthogonal and are called *Bender-Dunne polynomials*.

Relationships between the spectral theory and differential Galois theory have been studied by V. Spiridonov [97], F. Beukers [16] and Braverman et. al. [19]. As far as we know, Spiridonov was the first author that considered the usefulness of the Picard-Vessiot theory in the context of the quantum mechanics. This thesis agrees with his point of view.

Chapter 1

Theoretical Background

In this chapter we set the main theoretical background needed to understand the results of this thesis. We start setting conventions and notations that will be used along this work.

- The sets $\mathbb{Z}_+$, $\mathbb{Z}_-$, $\mathbb{Z}_+^*$ and $\mathbb{Z}_-^*$ are defined as

  $$\mathbb{Z}_+ = \{n \in \mathbb{Z} : \quad n \geq 0\}, \quad \mathbb{Z}_- = \{n \in \mathbb{Z} : \quad n \leq 0\}, \quad \mathbb{Z}_+^* = \mathbb{Z}^+, \quad \mathbb{Z}_-^* = \mathbb{Z}^-.$$

- The cardinality of the set A will be denoted by $\mathrm{Card}(A)$.
- The determinant of the matrix A will be denoted by $\det A$.
- The set of matrices $n \times n$ with entries in $\mathbb{C}$ and determinant non-null, the general linear group over $\mathbb{C}$, will be denoted by $\mathrm{GL}(n, \mathbb{C})$.
- The derivation $d/d\xi$ will be denoted by ∂_ξ. For example, the derivations $' = d/dx$ and $\dot{} = d/dt$ are denoted by ∂_x and ∂_t respectively.

1.1 Picard-Vessiot theory

Picard-Vessiot theory is the Galois theory of linear differential equations. In the classical Galois theory, the main object is a group of permutations of the roots, while in the Picard-Vessiot theory it is a linear algebraic group. For polynomial equations we want a solution in terms of radicals, which from classical Galois theory it is well if the Galois group is a solvable group.

An analogous situation holds for linear homogeneous differential equations (see [15, 26, 67, 81]). The following definition is true in general dimension, but for simplicity we are restricting to matrices 2×2.

1.1.1 Definitions and Known Results

Definition 1.1.1. An algebraic group of matrices 2×2 is a subgroup $G\subset \mathrm{GL}(2,\mathbb{C})$, defined by algebraic equations in its matrix elements and in the inverse of its determinant. That is, for $A\in \mathrm{GL}(2,\mathbb{C})$ given by

$$A=\begin{pmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix},\quad \det A = x_{11}x_{22}-x_{21}x_{22}$$

there exists a set of polynomials

$$\{P_i(x_{11},x_{12},x_{21},x_{22},1/\det A)\}_{i\in I},$$

such that

$$A\in G \quad \Leftrightarrow \quad \forall i\in I, P_i(x_{11},x_{12},x_{21},x_{22},1/\det A)=0.$$

In this case we say that G is an algebraic manifold endowed with a group structure.

Examples (Known algebraic groups). The following algebraic groups should be kept in mind throughout this work.

- Special linear group group:
  $\mathrm{SL}(2,\mathbb{C})=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix},\quad ad-bc=1,\quad a,b,c,d\in\mathbb{C}\right\}$
- Borel group: $\mathbb{B}=\mathbb{C}^*\ltimes\mathbb{C}=\left\{\begin{pmatrix} c & d\\ 0 & c^{-1}\end{pmatrix},\quad c\in\mathbb{C}^*,\quad d\in\mathbb{C}\right\}$
- Multiplicative group: $\mathbb{G}_m=\left\{\begin{pmatrix} c & 0\\ 0 & c^{-1}\end{pmatrix},\quad c\in\mathbb{C}^*\right\}$
- Additive group: $\mathbb{G}_a=\left\{\begin{pmatrix} 1 & d\\ 0 & 1\end{pmatrix},\quad d\in\mathbb{C}\right\}$
- Infinite dihedral group (also called meta-abelian group):
  $\mathbb{D}_\infty=\left\{\begin{pmatrix} c & 0\\ 0 & c^{-1}\end{pmatrix},\quad c\in\mathbb{C}^*\right\}\cup\left\{\begin{pmatrix} 0 & d\\ -d^{-1} & 0\end{pmatrix},\quad d\in\mathbb{C}^*\right\}$
- $n-$quasi-roots: $\mathbb{G}^{\{n\}}=\left\{\begin{pmatrix} c & d\\ 0 & c^{-1}\end{pmatrix},\quad c^n=1,\quad d\in\mathbb{C}\right\}$
- $n-$roots: $\mathbb{G}^{[n]}=\left\{\begin{pmatrix} c & 0\\ 0 & c^{-1}\end{pmatrix},\quad c^n=1\right\}$
- Identity group: $e=\left\{\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\right\}$

- The tetrahedral group $A_4^{\mathrm{SL}_2}$ of order 24 is generated by matrices

$$M_1 = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad \text{and} \quad M_2 = \frac{1}{3}(2\xi - 1)\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix},$$

where ξ denotes a primitive sixth root of unity, that is, $\xi^2 - \xi + 1 = 0$.

- The octahedral group $S_4^{\mathrm{SL}_2}$ of order 48 is generated by matrices

$$M_1 = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad \text{and} \quad M_2 = \frac{1}{2}\xi(\xi^2 + 1)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

where ξ denotes a primitive eighth root of unity, that is, $\xi^4 + 1 = 0$.

- The icosahedral group $A_5^{\mathrm{SL}_2}$ of order 120 is generated by matrices

$$M_1 = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad \text{and} \quad M_2 = \frac{1}{5}\begin{pmatrix} \phi & \psi \\ \psi & -\phi \end{pmatrix},$$

where ξ denotes a primitive tenth root of unity, that is, $\xi^4 - \xi^3 + \xi^2 - \xi + 1 = 0$, $\phi = \xi^3 - \xi^2 + 4\xi - 2$ and $\psi = \xi^3 + 3\xi^2 - 2\xi + 1$.

Recall that a group G is called solvable if and only if there exists a chain of normal subgroups

$$e = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G$$

such that the quotient G_i/G_j is abelian for all $n \geq i \geq j \geq 0$. Also recall that an algebraic group G has a unique connected normal algebraic subgroup G^0 of finite index. This means that the identity connected component G^0 is the largest connected algebraic subgroup of G containing the identity. For instance, if $G = G^0$, we say that G is a *connected group*.

Furthermore, if G^0 satisfy some property, then we say that G virtually satisfy such property. In this way, virtually solvability of G means solvability of G^0 and virtually abelianity of G means abelianity of G^0 (see [106]).

Theorem 1.1.2 (Lie-Kolchin). *Let $G \subseteq \mathrm{GL}(2,\mathbb{C})$ be a virtually solvable group. Then G^0 is triangularizable, that is, conjugate to a subgroup of upper triangular matrices.*

Definition 1.1.3. Let $G \subseteq \mathrm{GL}(2,\mathbb{C})$ be a group acting on a vector space V. We say that (the action of) G is either:

1. *Reducible*, if there exists a non-trivial subspace $W \subset V$ such that $G(W) \subset W$. We say that G is *irreducible* if G is not reducible.

2. *Imprimitive*, if G is irreducible and there exists subspaces V_i such that $V = V_1 \otimes \cdots \otimes V_m$, where G permutes transitively the V_i, i.e $\forall i = 1, \ldots, m$, $\forall g \in G$, $\exists j \in \{1, \ldots, m\}$ such that $g(V_i) = V_j$. We say that $V_1, \ldots, V_m$ form a system of *imprimitivity* for G.

3. *Primitive*, if G is irreducible and not imprimitive.

Examples. Any subgroup of the Borel group is reducible, the infinite dihedral group is imprimitive and the groups A^{SL_2}, $S_4^{\mathrm{SL}_2}$, $A_5^{\mathrm{SL}_2}$, $\mathrm{SL}(2,\mathbb{C})$ are primitives (see [81, 106]).

Definition 1.1.4 (Differential Fields)**.** Let K (depending on x) be a commutative field of characteristic zero, ∂_x a derivation, that is, a map $\partial_x : K \to K$ satisfying $\partial_x(a+b) = \partial_x a + \partial_x b$ and $\partial_x(ab) = \partial_x a \cdot b + a \cdot \partial_x b$ for all $a, b \in K$. By $\mathcal{C}$ we denote the field of constants of K

$$\mathcal{C} = \{c \in K | \partial_x c = 0\}$$

which is also of characteristic zero and will be assumed algebraically closed. In this terms, we say that K is a *differential field* with the derivation ∂_x.

Along this work, up to some specifications, *we consider as differential field the smallest differential containing the coefficients*. Furthermore, up to special considerations, we analyze second order linear homogeneous differential equations, that is, equations in the form

$$\mathcal{L} := \partial_x^2 y + a\partial_x y + by = 0, \quad a, b \in K. \tag{1.1}$$

Definition 1.1.5 (Picard-Vessiot Extension)**.** Suppose that y_1, y_2 is a basis of solutions of $\mathcal{L}$ given in equation (1.1), i.e., y_1, y_2 are linearly independent over K and every solution is a linear combination over $\mathcal{C}$ of these two. Let $L = K\langle y_1, y_2\rangle = K(y_1, y_2, \partial_x y_1, \partial_x y_2)$ the differential extension of K such that $\mathcal{C}$ is the field of constants for K and L. In this terms, we say that L, the smallest differential field containing K and $\{y_1, y_2\}$, is the *Picard-Vessiot extension* of K for $\mathcal{L}$.

Definition 1.1.6 (Differential Galois Groups)**.** Assume K, L and $\mathcal{L}$ as in previous definition. The group of all differential automorphisms (automorphisms that commutes with derivation) of L over K is called the *differential Galois group* of L over K and is denoted by $\mathrm{DGal}(L/K)$. This means that for $\sigma \in \mathrm{DGal}(L/K)$, $\sigma(\partial_x a) = \partial_x(\sigma(a))$ for all $a \in L$ and $\forall a \in K$, $\sigma(a) = a$.

Assume that $\{y_1, y_2\}$ is a fundamental system of solutions (basis of solutions) of $\mathcal{L}$. If $\sigma \in \mathrm{DGal}(L/K)$ then $\{\sigma y_1, \sigma y_2\}$ is another fundamental system of $\mathcal{L}$. Hence there exists a matrix

$$A_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C}),$$

such that

$$\sigma \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sigma(y_1) \\ \sigma(y_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} A_\sigma,$$

in a natural way, we can extend to systems:

$$\sigma \begin{pmatrix} y_1 & y_2 \\ \partial_x y_1 & \partial_x y_2 \end{pmatrix} = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \\ \sigma(\partial_x y_1) & \sigma(\partial_x y_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ \partial_x y_1 & \partial_x y_2 \end{pmatrix} A_\sigma.$$

This defines a faithful representation $\mathrm{DGal}(L/K) \to \mathrm{GL}(2,\mathbb{C})$ and it is possible to consider $\mathrm{DGal}(L/K)$ as a subgroup of $\mathrm{GL}(2,\mathbb{C})$. It depends on the choice of the fundamental system $\{y_1, y_2\}$, but only up to conjugacy.

One of the fundamental results of the Picard-Vessiot theory is the following theorem (see [53, 55]).

Theorem 1.1.7. *The differential Galois group* $\mathrm{DGal}(L/K)$ *is an algebraic subgroup of* $\mathrm{GL}(2,\mathbb{C})$.

Examples. Consider the following differential equations:

- $\mathcal{L} := \partial_x^2 y = 0$, the basis of solutions is given by $y_1 = 1$, $y_2 = x$. If we set as differential field $K = \mathbb{C}(x)$, we can see that $\sigma(1) = 1$, $\sigma(x) = x$, then the Picard-Vessiot extension $L = K$ and for instance $\mathrm{DGal}(L/K) = e$:

$$\sigma \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Now, if we set $K = \mathbb{C}$, then $L = K\langle x \rangle$, $\partial_x x \in \mathbb{C}$, $\partial_x(\sigma(x)) = \sigma(\partial_x x) = \sigma(1) = 1 = \partial_x x$, so $\sigma(x) = x + d$, $d \in \mathbb{C}$ and for instance $\mathrm{DGal}(L/K) = \mathbb{G}_a$:

$$\sigma \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y_1 \\ dy_1 + y_2 \end{pmatrix}.$$

- $\mathcal{L} := \partial_x^2 y = \kappa y$, $\kappa \in \mathbb{C}^*$, the basis of solutions is given by $y_1 = e^{\sqrt{\kappa}x}$, $y_2 = e^{-\sqrt{\kappa}x}$, with $\kappa \neq 0$. If we set as differential field $K = \mathbb{C}(x)$, we can see that $L = K\langle e^{\sqrt{k}x} \rangle = K(e^{\sqrt{k}x})$,

$$\sigma \left(\frac{\partial_x y_1}{y_1} \right) = \frac{\partial_x(\sigma(y_1))}{\sigma(y_1)} = \frac{\partial_x y_1}{y_1}, \quad \sigma \left(\frac{\partial_x y_2}{y_2} \right) = \frac{\partial_x(\sigma(y_2))}{\sigma(y_2)} = \frac{\partial_x y_2}{y_2},$$

$\sigma(y_1 y_2) = \sigma(y_1)\sigma(y_2) = y_1 y_2 = 1$, $\sigma(y_1) = cy_1$, $\sigma(y_2) = dy_2$, $c, d \in \mathbb{C}$, but $cd = 1$ and for instance $\mathrm{DGal}(L/K) = \mathbb{G}_m$:

$$\sigma \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} = \begin{pmatrix} cy_1 \\ c^{-1} y_2 \end{pmatrix},$$

Now, if we set $K = \mathbb{C}$, we obtain the same result.

- $\mathcal{L} := \partial_x^2 y + \frac{n-1}{nx} \partial_x y = 0$, the basis of solutions is given by $y_1 = z$, where $z^n = x$, $y_2 = 1$. If we set $K = \mathbb{C}(x)$, then $L = K \left\langle x^{\frac{1}{n}} \right\rangle$, $y_1^n = x \in \mathbb{C}(x)$, $\sigma^n(y_1) = \sigma(y_1^n) = x$, $\sigma(y_1) = cy_1$, so that $c^n = 1$ and for instance $\mathrm{DGal}(L/K)$ is given by:

$$\sigma \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} cy_1 \\ y_2 \end{pmatrix}, \quad c^n = 1.$$

- $\mathcal{L} := \partial_x^2 y + \frac{n^2-1}{4n^2x^2}y = 0$, $n \in \mathbb{Z}$, the basis of solutions is given by $y_1 = x^{\frac{n+1}{2n}}$, $y_2 = x^{\frac{n-1}{2n}}$. If we set $K = \mathbb{C}(x)$ and n even, then $L = K\left\langle x^{\frac{1}{2n}}\right\rangle$,

$$\sigma(y_1) = cy_1, \quad \sigma^{2n}(y_1) = c^{2n}y_1^{2n} = \sigma(y_1^{2n}) = y_1^{2n}, \quad c^{2n} = 1,$$

$$\sigma(y_2) = dy_2, \quad \sigma^{2n}(y_2) = d^{2n}y_2^{2n} = \sigma(y_2^{2n}) = y_2^{2n}, \quad d^{2n} = 1,$$

  $\sigma(y_1y_2) = y_1y_2 = \sigma(y_1)\sigma(y_2) = cdy_1y_2$ so that $cd = 1$ and for instance $\mathrm{DGal}(L/K) = \mathbb{G}^{[2n]}$:

$$\sigma\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} = \begin{pmatrix} cy_1 \\ c^{-1}y_2 \end{pmatrix}, \quad c^{2n} = 1, \quad n > 1.$$

  Now, if we consider n odd, then $L = K\left\langle x^{\frac{1}{n}}\right\rangle$, and $\mathrm{DGal}(L/K) = \mathbb{G}^{[2n]}$.

- Cauchy-Euler equation

$$\mathcal{L} := \partial_x^2 y = \frac{m(m+1)}{x^2}y, \quad m \in \mathbb{C},$$

  the basis of solutions is $y_1 = x^{m+1}$, $y_2 = x^{-m}$. Setting $K = \mathbb{C}(x)$, we have the following possible cases:

  - for $m \in \mathbb{Z}$, $L = K$ and $\mathrm{DGal}(L/K) = e$,
  - for $m \in \mathbb{Q} \setminus \mathbb{Z}$, $L = K(x^m)$ and $\mathrm{DGal}(L/K) = \mathbb{G}^{[d]}$, where $m = n/d$,
  - for $m \in \mathbb{C} \setminus \mathbb{Q}$, $L = K(x^m)$ and $\mathrm{DGal}(L/K) = \mathbb{G}_m$.

Definition 1.1.8 (Integrability)**.** Consider the linear differential equation $\mathcal{L}$ such as in equation (1.1). We say that $\mathcal{L}$ is *integrable* if the Picard-Vessiot extension $L \supset K$ is obtained as a tower of differential fields $K = L_0 \subset L_1 \subset \cdots \subset L_m = L$ such that $L_i = L_{i-1}(\eta)$ for $i = 1, \ldots, m$, where either

1. η is *algebraic* over L_{i-1}, that is η satisfies a polynomial equation with coefficients in L_{i-1}.
2. η is *primitive* over L_{i-1}, that is $\partial_x\eta \in L_{i-1}$.
3. η is *exponential* over L_{i-1}, that is $\partial_x\eta/\eta \in L_{i-1}$.

We recall that the differential field of coefficients has been fixed before, i.e., the smallest differential field containing the coefficients.

We remark that the usual terminology in differential algebra for integrable equations is that the corresponding Picard-Vessiot extensions are called *Liouvillian*.

Theorem 1.1.9 (Kolchin)**.** *The equation* $\mathcal{L}$ *given in* (1.1) *is integrable if and only if* $\mathrm{DGal}(L/K)$ *is virtually solvable.*

Consider the differential equation

$$\mathcal{L} := \partial_x^2 \zeta = r\zeta, \quad r \in K. \tag{1.2}$$

We recall that equation (1.2) can be obtained from equation (1.1) through the change of variable

$$y = e^{-\frac{1}{2}\int a}\zeta, \quad r = \frac{a^2}{4} + \frac{\partial_x a}{2} - b \tag{1.3}$$

and equation (1.2) is called the *reduced form* of equation (1.1).

On the other hand, introducing the change of variable $v = \partial_x\zeta/\zeta$ we get the associated Riccati equation to equation (1.2)

$$\partial_x v = r - v^2, \quad v = \frac{\partial_x \zeta}{\zeta}, \tag{1.4}$$

where r is obtained by equation (1.3).

Theorem 1.1.10 (Singer 1981, [91])**.** *The Riccatti equation* (1.4) *has one algebraic solution over the differential field* K *if and only if the differential equation* (1.2) *is integrable.*

For $\mathcal{L}$ given by equation (1.2), it is very well known (see [53, 55, 81]) that $\mathrm{DGal}_K(\mathcal{L})$ is an algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$. The well known classification of subgroups of $\mathrm{SL}(2,\mathbb{C})$ (see [53, p.31], [56, p.7,27]) is the following.

Theorem 1.1.11. *Let* G *be an algebraic subgroup of* $\mathrm{SL}(2,\mathbb{C})$*. Then, up to conjugation, one of the following cases occurs.*

1. $G \subseteq \mathbb{B}$ *and then* G *is reducible and triangularizable.*
2. $G \nsubseteq \mathbb{B}$, $G \subseteq \mathbb{D}_\infty$ *and then* G *is imprimitive.*
3. $G \in \{A_4^{\mathrm{SL}_2}, S_4^{\mathrm{SL}_2}, A_5^{\mathrm{SL}_2}\}$ *and then* G *is primitive (finite)*
4. $G = \mathrm{SL}(2,\mathbb{C})$ *and then* G *is primitive (infinite).*

Definition 1.1.12. Consider the differential equation $\mathcal{L}$ given by equation (1.2). Let $\{\zeta_1, \zeta_2\}$ be a fundamental system of $\mathcal{L}$. Let $f = f(Y_1, Y_2) \in \mathcal{C}[Y_1, Y_2]$ be a homogeneous polynomial, we say that:

1. The polynomial f is an *invariant* with respect to $\mathcal{L}$ if its evaluation on a $\mathcal{C}$-basis $\{\zeta_1, \zeta_2\}$ of solutions is invariant under the action of $\mathrm{DGal}(L/K)$, that is, for every $\sigma \in \mathrm{DGal}(L/K)$, $\sigma h(x) = h(x)$, where $h(x) = f(\zeta_1(x), \zeta_2(x)) \in K$. The function $h(x)$ is called the *value* of the invariant polynomial f.

2. The polynomial f is a *semi-invariant* with respect to $\mathcal{L}$ if the logarithmic derivative $\frac{\partial_x h}{h}$ of its evaluation $h(x) = f(\zeta_1(x), \zeta_2(x))$ on any $\mathcal{C}$-basis $\{\zeta_1, \zeta_2\}$ is an element of K, that is, for every $\sigma \in \mathrm{DGal}(L/K)$, $\sigma\theta = \theta$, where $\theta = \partial_x h(x)/h(x) \in K$.

Theorem 1.1.13 (Kovacic, [56]). *Let $\{\zeta_1, \zeta_2\}$ be a fundamental system of solutions of $\mathcal{L}$ given by the differential equation (1.2). Then, for some $i \in \{1, 2\}$ and for every $\sigma \in \mathrm{DGal}(L/K)$, exclusively one of the following cases holds.*

1. $\mathrm{DGal}(L/K)$ *is reducible and then $f = Y_i$ is semi-invariant with respect to $\mathcal{L}$, i.e $\partial_x(\ln \zeta_i) \in K$.*
2. $\mathrm{DGal}(L/K)$ *is imprimitive and then $f_1 = \zeta_1\zeta_2$ is semi-invariant with respect to $\mathcal{L}$, $f_2 = (Y_1Y_2)^2$ is invariant with respect to $\mathcal{L}$, i.e $[K\langle\partial_x(\ln \zeta_i)\rangle : K] = 2$.*
3. $\mathrm{DGal}(L/K)$ *is finite primitive and then the invariants with respect to $\mathcal{L}$ is either $f_1 = (Y_1^4 + 8Y_1Y_2^3)^3$, or $f_2 = (Y_1^5Y_2 - Y_1Y_2^5)^2$ or $f_3 = Y_1^{11}Y_2 - 11Y_1^6Y_2^6 - Y_1Y_2^{11}$, i.e $[K\langle\partial_x(\ln \zeta_i)\rangle : K] = 4, 6, 12$.*
4. $\mathrm{DGal}(L/K)$ *is infinite primitive, i.e there are no non-trivial semi-invariants.*

Statements and proofs of theorems 1.1.9, 1.1.10 and 1.1.11 can be found in [81].

1.1.2 Kovacic's Algorithm

Considering $K = \mathbb{C}(x)$, $\mathcal{C} = \mathbb{C}$ in theorems 1.1.9, 1.1.11 and 1.1.13, Kovacic in 1986 ([56]) introduced an algorithm to solve the differential equation (1.2) showing that (1.2) is integrable if and only if the solution of the Riccati equation (1.4) is a rational function (case 1), is a root of polynomial of degree two (case 2) or is a root of polynomial of degree 4, 6, or 12 (case 3). For more details see reference [56]. Improvements for this algorithm are given in references [33, 34, 101]. Here, we follow the original version given by Kovacic in reference [56] with an adapted version presented in reference [5].

Each case in Kovacic's algorithm is related with each one of the algebraic subgroups of $\mathrm{SL}(2, \mathbb{C})$ and the associated Riccatti equation

$$\partial_x v = r - v^2 = \left(\sqrt{r} - v\right)\left(\sqrt{r} + v\right), \quad v = \frac{\partial_x \zeta}{\zeta}.$$

According to Theorem 1.1.11, there are four cases in Kovacic's algorithm. Only for cases 1, 2 and 3 we can solve the differential equation, but for the case 4 the differential equation is not integrable. It is possible that Kovacic's algorithm can provide us only one solution (ζ_1), so that we can obtain the second solution (ζ_2) through

$$\zeta_2 = \zeta_1 \int \frac{dx}{\zeta_1^2}. \tag{1.5}$$

Notations. For the differential equation given by

$$\partial_x^2\zeta = r\zeta, \qquad r = \frac{s}{t}, \quad s,t \in \mathbb{C}[x],$$

we use the following notations.

1. Denote by Γ' be the set of (finite) poles of r, $\Gamma' = \{c \in \mathbb{C} : t(c) = 0\}$.
2. Denote by $\Gamma = \Gamma' \cup \{\infty\}$.
3. By the order of r at $c \in \Gamma'$, $\circ(r_c)$, we mean the multiplicity of c as a pole of r.
4. By the order of r at ∞, $\circ(r_\infty)$, we mean the order of ∞ as a zero of r. That is $\circ(r_\infty) = \deg(t) - \deg(s)$.

The four cases

Case 1. In this case $[\sqrt{r}]_c$ and $[\sqrt{r}]_\infty$ means the Laurent series of $\sqrt{r}$ at c and the Laurent series of $\sqrt{r}$ at ∞ respectively. Furthermore, we define $\varepsilon(p)$ as follows: if $p \in \Gamma$, then $\varepsilon(p) \in \{+,-\}$. Finally, the complex numbers $\alpha_c^+, \alpha_c^-, \alpha_\infty^+, \alpha_\infty^-$ will be defined in the first step. If the differential equation has no poles it only can fall in this case.

Step 1. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

(c_0) If $\circ(r_c) = 0$, then

$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = 0.$$

(c_1) If $\circ(r_c) = 1$, then

$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = 1.$$

(c_2) If $\circ(r_c) = 2$, and

$$r = \cdots + b(x-c)^{-2} + \cdots, \quad \text{then}$$

$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = \frac{1 \pm \sqrt{1+4b}}{2}.$$

(c_3) If $\circ(r_c) = 2v \geq 4$, and

$$r = (a(x-c)^{-v} + ... + d(x-c)^{-2})^2 + b(x-c)^{-(v+1)} + \cdots, \quad \text{then}$$

$$\left[\sqrt{r}\right]_c = a(x-c)^{-v} + ... + d(x-c)^{-2}, \quad \alpha_c^\pm = \frac{1}{2}\left(\pm\frac{b}{a} + v\right).$$

(∞_1) If $\circ(r_\infty) > 2$, then

$$\left[\sqrt{r}\right]_\infty = 0, \quad \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1.$$

(∞_2) If $\circ(r_\infty) = 2$, and $r = \cdots + bx^2 + \cdots$, then

$$\left[\sqrt{r}\right]_\infty = 0, \quad \alpha_\infty^\pm = \frac{1 \pm \sqrt{1+4b}}{2}.$$

(∞_3) If $\circ(r_\infty) = -2v \leq 0$, and

$$r = (ax^v + ... + d)^2 + bx^{v-1} + \cdots, \quad \text{then}$$

$$\left[\sqrt{r}\right]_\infty = ax^v + ... + d, \quad and \quad \alpha_\infty^\pm = \frac{1}{2}\left(\pm\frac{b}{a} - v\right).$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ n \in \mathbb{Z}_+ : n = \alpha_\infty^{\varepsilon(\infty)} - \sum_{c\in\Gamma'} \alpha_c^{\varepsilon(c)}, \forall\left(\varepsilon\left(p\right)\right)_{p\in\Gamma} \right\}.$$

If $D = \emptyset$, then we should start with the case 2. Now, if $\mathrm{Card}(D) > 0$, then for each $n \in D$ we search $\omega \in \mathbb{C}(x)$ such that

$$\omega = \varepsilon\left(\infty\right)\left[\sqrt{r}\right]_\infty + \sum_{c\in\Gamma'} \left(\varepsilon\left(c\right)\left[\sqrt{r}\right]_c + \alpha_c^{\varepsilon(c)}(x-c)^{-1} \right).$$

Step 3. For each $n \in D$, search for a monic polynomial P_n of degree n with

$$\partial_x^2 P_n + 2\omega\partial_x P_n + (\partial_x\omega + \omega^2 - r)P_n = 0. \tag{1.6}$$

If success is achieved then $\zeta_1 = P_n e^{\int \omega}$ is a solution of the differential equation. Else, case 1 cannot hold.

Case 2. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c \in \Gamma'$ and ∞ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Gamma'$ and for ∞ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

(c_1) If $\circ(r_c) = 1$, then $E_c = \{4\}$

(c_2) If $\circ(r_c) = 2$, and $r = \cdots + b(x-c)^{-2} + \cdots$, then

$$E_c = \left\{ 2 + k\sqrt{1+4b} : k = 0, \pm 2 \right\}.$$

(c_3) If $\circ(r_c) = v > 2$, then $E_c = \{v\}$

(∞_1) If $\circ(r_\infty) > 2$, then $E_\infty = \{0, 2, 4\}$

(∞_2) If $\circ(r_\infty) = 2$, and $r = \cdots + bx^2 + \cdots$, then

$$E_\infty = \left\{2 + k\sqrt{1+4b} : k = 0, \pm 2\right\}.$$

(∞_3) If $\circ(r_\infty) = v < 2$, then $E_\infty = \{v\}$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ n \in \mathbb{Z}_+ : \quad n = \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma'} e_c\right), \forall e_p \in E_p, \quad p \in \Gamma \right\}.$$

If $D = \emptyset$, then we should start the case 3. Now, if $\mathrm{Card}(D) > 0$, then for each $n \in D$ we search a rational function θ defined by

$$\theta = \frac{1}{2}\sum_{c\in\Gamma'} \frac{e_c}{x-c}.$$

Step 3. For each $n \in D$, search a monic polynomial P_n of degree n, such that

$$\partial_x^3 P_n + 3\theta\partial_x^2 P_n + (3\partial_x\theta + 3\theta^2 - 4r)\partial_x P_n + \left(\partial_x 2\theta + 3\theta\partial_x\theta + \theta^3 - 4r\theta - 2\partial_x r\right) P_n = 0. \tag{1.7}$$

If P_n does not exist, then case 2 cannot hold. If such a polynomial is found, set $\phi = \theta + \partial_x P_n / P_n$ and let ω be a solution of

$$\omega^2 + \phi\omega + \frac{1}{2}\left(\partial_x\phi + \phi^2 - 2r\right) = 0.$$

Then $\zeta_1 = e^{\int \omega}$ is a solution of the differential equation.

Case 3. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c \in \Gamma'$ and ∞ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Gamma'$ and for ∞ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

(c_1) If $\circ(r_c) = 1$, then $E_c = \{12\}$

(c_2) If $\circ(r_c) = 2$, and $r = \cdots + b(x-c)^{-2} + \cdots$, then

$$E_c = \left\{6 + k\sqrt{1+4b} : \quad k = 0, \pm1, \pm2, \pm3, \pm4, \pm5, \pm6\right\}.$$

(∞) If $\circ(r_\infty) = v \geq 2$, and $r = \cdots + bx^2 + \cdots$, then

$$E_\infty = \left\{6 + \frac{12k}{m}\sqrt{1+4b} : \; k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\right\}, \; m \in \{4, 6, 12\}.$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ n \in \mathbb{Z}_+ : \quad n = \frac{m}{12}\left(e_\infty - \sum_{c \in \Gamma'} e_c\right), \forall e_p \in E_p, \quad p \in \Gamma \right\}.$$

In this case we start with $m = 4$ to obtain the solution, afterwards $m = 6$ and finally $m = 12$. If $D = \emptyset$, then the differential equation is not integrable because it falls in the case 4. Now, if $\mathrm{Card}(D) > 0$, then for each $n \in D$ with its respective m, search a rational function

$$\theta = \frac{m}{12}\sum_{c \in \Gamma'} \frac{e_c}{x - c}$$

and a polynomial S defined as

$$S = \prod_{c \in \Gamma'} (x - c).$$

Step 3. Search for each $n \in D$, with its respective m, a monic polynomial $P_n = P$ of degree n, such that its coefficients can be determined recursively by

$$P_{-1} = 0, \quad P_m = -P,$$

$$P_{i-1} = -S\partial_x P_i - ((m-i)\,\partial_x S - S\theta)\,P_i - (m-i)\,(i+1)\,S^2 r P_{i+1},$$

where $i \in \{0, 1 \ldots, m-1, m\}$. If P does not exist, then the differential equation is not integrable because it falls in Case 4. Now, if P exists search ω such that

$$\sum_{i=0}^{m} \frac{S^i P}{(m-i)!}\omega^i = 0,$$

then a solution of the differential equation is given by

$$\zeta = e^{\int \omega},$$

where ω is solution of the previous polynomial of degree m.

Remark 1.1.14 ([5]). If the differential equation falls only in the case 1 of Kovacic's algorithm, then its differential Galois group is given by one of the following groups:

I1 e when the algorithm provides two rational solutions.

I2 $\mathbb{G}^{[n]}$ when the algorithm provides two algebraic solutions ζ_1, ζ_2 such that $\zeta_1^n, \zeta_2^n \in \mathbb{C}(x)$ and $\zeta_1^{n-1}, \zeta_2^{n-1} \notin \mathbb{C}(x)$.

I3 $\mathbb{G}^{\{n\}}$ when the algorithm provides only one algebraic solution ζ such that $\zeta^n \in \mathbb{C}(x)$ with n minimal.

I4 $\mathbb{G}_m$ when the algorithm provides two non-algebraic solutions.

I5 $\mathbb{G}_a$ when the algorithm provides one rational solution and the second solution is not algebraic.

I6 $\mathbb{B}$ when the algorithm only provides one solution ζ such that ζ and its square are not rational functions.

Kovacic's Algorithm in Maple

In order to analyze second order linear differential equations with rational coefficients, generally without parameters, a standard procedure is using MAPLE, and especially commands `dsolve` and `kovacicsols`. Whenever the command `kovacicsols` yields an output "`[ ]`", it means that the second order linear differential equation being considered is not integrable, and thus its Galois group is non-virtually solvable.

In some cases, moreover, `dsolve` makes it possible to obtain the solutions in terms of special functions such as *Airy functions*, *Bessel functions* and **hypergeometric functions**, among others (see [1]).

There is a number of second order linear equations whose coefficients are not rational, and whose solutions MAPLE can find with the command `dsolve` but the presentation of the solutions is very complicated, furthermore the command `kovacicsols` does not work with such coefficients. These problems, in some cases, can be solved by our *algebrization method* (see section 2.3.1 and see also [5]).

Beyond Kovacic's Algorithm

According to the works of Michael Singer and Felix Ulmer [91, 94, 95, 99, 100], we can have another perspective of the Kovacic's algorithm by means of the m-th *symmetric power* of a linear differential equation $\mathcal{L}$, which is denoted as $\mathcal{L}^{\circledS m}$.

Theorem 1.1.15. *Let $\mathcal{L}$ be a linear homogeneous differential equation of arbitrary order n. For any $m \geq 1$ there is another linear homogeneous differential equation, denoted by $\mathcal{L}^{\circledS m}$, with the following property. If $\zeta_1, \ldots, \zeta_n$ are any solutions of $\mathcal{L}$ then any homogeneous polynomial in $\zeta_1, \ldots, \zeta_n$ of degree m is a solution of $\mathcal{L}^{\circledS m}$.*

In this way, Kovacic's algorithm can be stated as follows (see also [33, 34, 101]).

Algorithm. Consider $\mathcal{L} = \partial_x^2\zeta - r\zeta$.

Step 1. Check if $\mathcal{L}$ has an exponential solution $\zeta = e^{\int u}$ (with $u \in \mathbb{C}(x)$). If so then $\zeta = e^{\int u}$.

Step 2. Check if $\mathcal{L}^{ⓢ 2}$ has an exponential solution $\zeta = e^{\int u}$ (with $u \in \mathbb{C}(x)$). If so, let v be a root of

$$v^2 + uv + \left(\frac{1}{2}\partial_x u + \frac{1}{2}u^2 - u\right) = 0.$$

Then $\zeta = e^{\int v}$.

Step 3. Check if $\mathcal{L}^{ⓢ 4}$, $\mathcal{L}^{ⓢ 6}$, or $\mathcal{L}^{ⓢ 12}$ has an exponential solution $\zeta = e^{\int u}$ (with $u \in \mathbb{C}(x)$). If so, then there is a polynomial of degree 4, 6 or 12 (respectively) such that if v is a solution of it then $\zeta = e^{\int v}$.

This algorithm can be generalized using the results of Michael Singer in [91]. The trick is to find the correct numbers (like $2, 4, 6, 12$ of the Kovacic's algorithm).

Theorem 1.1.16. *Suppose a linear homogeneous differential equation of order n is integrable. Then it has a solution of the form*

$$\zeta = e^{\int v}$$

where v is algebraic over $\mathbb{C}(x)$. The degree of v is bounded by $I(n)$, which is defined inductively by

$$\begin{aligned} I(0) &= 1 \\ I(n) &= \max\{J(n),\ n!I(n-1)\} \\ J(n) &= \left(\sqrt{8n}+1\right)^{2n^2} - \left(\sqrt{8n}-1\right)^{2n^2}. \end{aligned}$$

The following theorem is due to Singer and Ulmer (see [95]).

Theorem 1.1.17 (Singer & Ulmer, [95])**.** *Let $\mathcal{L}$ be a homogeneous n-th order linear differential equation over K with differential Galois group $\mathrm{DGal}(L/K) \subseteq \mathrm{GL}(n, \mathcal{C})$.*

1. *If $\mathcal{L}$ has a Liouvillian solution whose logarithmic derivative is algebraic of degree m, then there is a $\mathrm{DGal}(L/K)$-semi-invariant of degree m in $\mathcal{C}[Y_1, \ldots, Y_n]$ which factors into linear forms.*

2. *If there is a $\mathrm{DGal}(L/K)$-semi-invariant of degree m in $\mathcal{C}[Y_1, \ldots, Y_n]$ which factors into linear forms, then $\mathcal{L}$ has a Liouvillian solution whose logarithmic derivative is algebraic of degree $\leq m$*

This algorithm is not considered implementable; the numbers $I(n)$ are simply much too large. For example $I(2) = 384064$, so this algorithm would require checking if $\mathcal{L}^{ⓢ m}$ has a solution ζ with $\partial_x\zeta/\zeta \in \mathbb{C}(x)$ for $m = 1, 2, \ldots, 384064$. However, specific algorithms exist for order three and higher, see [40, 95, 96, 99, 100, 46].

1.1.3 Eigenrings

We consider two different formalisms for Eigenrings, the matrix and operators formalism. We start with the Matrix formalism of Eigenrings following M. Barkatou in [7], but restricting again to 2×2 matrices.

Let K be a differential field and let A be a matrix in $\mathrm{GL}(2, K)$ such that,

$$\partial_x \mathbf{X} = -A\mathbf{X}. \tag{1.8}$$

Consider a matrix equation (1.8) and let $P \in \mathrm{GL}(2, K)$. The substitution $\mathbf{X} = P\mathbf{Y}$ leads to the matrix equation

$$\partial_x \mathbf{Y} = -B\mathbf{Y}, \quad B = P^{-1}(\partial_x P + AP). \tag{1.9}$$

Definition 1.1.18. The matrices A and B are *equivalents* over K, denoted by $A \sim B$, when there exists a matrix $P \in \mathrm{GL}(2, K)$ satisfying equation (1.9). The systems (1.8) and (1.9) are equivalent, denoted by $[A] \sim [B]$, when A and B are equivalent.

By equation (1.9), we have $PB = \partial_x P + AP$. In general, assuming $PB = PA$, where P is a 2×2 matrix, i.e, P is not necessarily in $\mathrm{GL}(2, K)$, we obtain $PA = \partial_x P + AP$, which leads us to the following definition.

Definition 1.1.19. The Eigenring of the system $[A]$, denoted by $\mathcal{E}(A)$, is the set of 2×2 matrices P in K satisfying

$$\partial_x P = PA - AP. \tag{1.10}$$

Equation (1.10) can be viewed as a system of 4 first-order linear differential equations over K. Thus, $\mathcal{E}(A)$ is a $\mathcal{C}-$vector space of finite dimension ≤ 4. Owing to the product of two elements of $\mathcal{E}(A)$ is also an element of $\mathcal{E}(A)$ and the identity matrix I_2 belongs to $\mathcal{E}(A)$, we have that $\mathcal{E}(A)$ is an algebra over $\mathcal{C}$, i.e., $\mathcal{E}(A)$ is a $\mathcal{C}-$algebra. As a consequence, we have the following results that can be found in [7].

Proposition 1.1.20 ([7])**.** *Any element P of $\mathcal{E}(A)$ has:*

- *a minimal polynomial with coefficients in $\mathcal{C}$ and*
- *all its eigenvalues are constant.*

Proposition 1.1.21 ([7])**.** *If two systems $[A]$ and $[B]$ are equivalent, their eigenrings $\mathcal{E}(A)$ and $\mathcal{E}(B)$ are isomorphic as $\mathcal{C}-$algebras. In particular, one has* $\dim_{\mathcal{C}} \mathcal{E}(A) = \dim_{\mathcal{C}} \mathcal{E}(B)$.

Definition 1.1.22. The system $[A]$ is called *reducible* when $A \sim B$, being B given by

$$B = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}.$$

When $[A]$ is reducible and $b_{21} = 0$, the system $[A]$ is called *decomposable* or *completely reducible*. The system $[A]$ is called *irreducible* or *indecomposable* when $[A]$ is not reducible.

Assume that the eigenring $\mathcal{E}(A)$ is known.

Theorem 1.1.23 ([7]). *If $\mathcal{E}(A)$ is not a division ring then $[A]$ is reducible and the reduction can be carried out by a matrix $P \in \mathrm{GL}(2, K)$ that can be computed explicitly.*

The condition $\mathcal{E}(A)$ *is not a division ring* implies $\dim_{\mathcal{C}} \mathcal{E}(A) > 1$. Indeed, if $P \in \mathcal{E}(A) \setminus \{0\}$ is not invertible, then the family $\{I, P\}$ is linearly independent (over $\mathcal{C}$) and hence $\dim_{\mathcal{C}} \mathcal{E}(A) > 1$. In our case the converse is true, due to the field of constants $\mathcal{C}$ is algebraically closed. Indeed, suppose that $\dim_{\mathcal{C}} \mathcal{E}(A) > 1$ then there exists $P \in \mathcal{E}(A)$ such that the family $\{I, P\}$ be linearly independent. Since $\mathcal{C}$ is algebraically closed, there exists $\lambda \in \mathcal{C}$ such that $\det(P - \lambda I) = 0$. Hence $\mathcal{E}(A)$ contains an element, namely $P - \lambda I$, which is non-zero and non invertible.

The computation of eigenrings of the system $[A]$ is implemented in *ISOLDE* (Integration of Systems of Ordinary Linear Differential Equations). The function is `eigenring`, the calling sequence is `eigenring(A, x)` being the parameters: A - a square rational function matrix with coefficients in an algebraic extension of the rational numbers and x - the independent variable (a name). ISOLDE was written in Maple V and it is available at `http://isolde.sourceforge.net/`.

In operators formalism, we restrict ourselves to second order differential operators and we follow the works of Singer, Barkatou and Van Hoeij (see [93, 7, 43, 44, 45]). A differential equation $\mathcal{L} := \partial_x^2 y + a\partial_x y + by = 0$ with $a, b \in K$ corresponds to a differential operator $f = \partial_x^2 + a\partial_x + b$ acting on y. The differential operator f is an element of the non-commutative ring $K[\partial_x]$.

The factorization of operators is very important to solve differential equations, that is, a factorization $f = \mathfrak{L}\mathfrak{R}$ where $\mathfrak{L}, \mathfrak{R} \in K[\partial_x]$ is useful for computing solutions of f because solutions of the right-hand factor $\mathfrak{R}$ are solutions of f as well.

Definition 1.1.24. Let $\mathfrak{L}$ be a second order differential operator, i.e $\mathcal{L} := \mathfrak{L}(y) = 0$. Denote $V(\mathfrak{L})$ as the solution space of $\mathcal{L}$. The *Eigenring* of $\mathfrak{L}$, denoted by $\mathcal{E}(\mathfrak{L})$, is the set of all operators $\mathfrak{R}$ for which $\mathfrak{R}(V(\mathfrak{L}))$ is a subset of $V(\mathfrak{L})$, that is $\mathfrak{L}\mathfrak{R} = \mathfrak{S}\mathfrak{L}$, where $\mathfrak{S}$ is also an operator.

As consequence of the previous definition, $\mathfrak{R}$ is an endomorphism of the solution space $V(\mathfrak{L})$. This means that we can think of $\mathfrak{R}$ as a linear map $V \to V$ and choosing one local basis of V we obtain, by linear algebra, that $\mathfrak{R}$ has a matrix $M_{\mathfrak{R}}$. The characteristic polynomial of this map can be computed with the classical methods of linear algebra. For endomorphisms $\mathfrak{R}$, the product of $\mathfrak{L}$ and $\mathfrak{R}$ is divisible on the right by $\mathfrak{L}$. This means that if $\mathfrak{L}(y) = 0$, then $\mathfrak{L}(\mathfrak{R}(y)) = 0$, so that $\mathfrak{R}$ map $V \to V$.

For the general case of operators

$$\mathfrak{L} = \sum_{k=0}^{n} b_k \partial_x^k, \quad \mathfrak{R} = \sum_{i=0}^{m} a_i \partial_x^i, \quad a_i, b_i \in K, \quad n > m, \quad \mathcal{L} := \mathfrak{L}(y) = 0$$

and $G = \mathrm{DGal}(L/K)$, we can see $\mathfrak{R}$ as a G-map. Now, denoting by P the characteristic polynomial of $\mathfrak{R}$, assume that there exists polynomials P_1, P_2 with $\gcd(P_1, P_2) = 1$ such that $P = P_1 P_2$. By Cayley-Hamilton theorem we have that $P(M_{\mathfrak{R}}) = 0$ and by kernel theorem we have that $V = \ker(P_1(M_{\mathfrak{R}})) \oplus \ker(P_2(M_{\mathfrak{R}}))$, in where $\ker(P_1(M_{\mathfrak{R}}))$ and $\ker(P_2(M_{\mathfrak{R}}))$ are invariants under G.

Let λ denote an eigenvalue of $\mathfrak{R}$; there exists a non-trivial eigenspace $V_\lambda \subseteq V$, which means that $\mathfrak{L}$ and $\mathfrak{R} - \lambda$ has common solutions and therefore right $-$ $\gcd(\mathfrak{L}, \mathfrak{R} - \lambda)$ is non-trivial, this means that right $- \gcd(\mathfrak{L}, \mathfrak{R} - \lambda)$ it is a factor of $\mathfrak{L}$.

Returning to the second order operators, we establish the relationship between the Eigenring of the system $[A]$ and the Eigenring of the operator $\mathfrak{L}$. We start recalling that $\mathcal{L}$ given by $\partial_x^2 y + a\partial_x y + by = 0$, $a, b \in K$, can be written as the system

$$\partial_x \begin{pmatrix} y \\ \partial_x y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} y \\ \partial_x y \end{pmatrix},$$

and the system of linear differential equations

$$\partial_x \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \quad a, b, c, d \in K,$$

by means of an elimination process, is equivalent to the second-order equation

$$\partial_x^2 y - \left(a + d + \frac{\partial_x b}{b} \right) \partial_x y - \left(\partial_x a + bc - ad - a\frac{\partial_x b}{b} \right) y = 0, \quad b \neq 0. \qquad (1.11)$$

In this way, we can go from operators to systems and reciprocally computing the Eigenrings in both formalism. In particular, we emphasize in the operator $\mathfrak{L} = \partial_x^2 + p\partial_x + q$, which is equivalent to the system $[A]$, where A is given by

$$A = \begin{pmatrix} 0 & -1 \\ q & p \end{pmatrix} \quad p, q \in K.$$

As immediate consequence we have the following lemmas.

Lemma 1.1.25. *Consider $\mathfrak{L}$, A and P as follows:*

$$\mathfrak{L} = \partial_x^2 + p\partial_x + q, \quad A = \begin{pmatrix} 0 & -1 \\ q & p \end{pmatrix}, \quad P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d, p, q \in K.$$

The following statements holds

1. *If $P \in \mathcal{E}(A)$, then $\mathfrak{R} = a + b\partial_x \in \mathcal{E}(\mathfrak{L})$.*
2. *If $\mathfrak{R} = a + b\partial_x \in \mathcal{E}(\mathfrak{L})$, then $P \in \mathcal{E}(A)$, where P is given by*

$$P = \begin{pmatrix} a & b \\ \partial_x a - bq & a + \partial_x b - bp \end{pmatrix}.$$

3. *$1 \leq \dim_{\mathcal{C}} \mathcal{E}(\mathfrak{L}) \leq 4$.*
4. *$P \in \mathrm{GL}(2, K) \Leftrightarrow \frac{\partial_x a}{a} - \frac{a}{b} + p \neq \frac{\partial_x b}{b} - \frac{b}{a} q$.*

Proof. Suppose that $V_{\mathfrak{L}}$ and V_A are solution spaces of $\mathcal{L}$ and $[A]$ respectively.

1. If $P \in \mathcal{E}(A)$, then for a solution $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ of $\partial_x Y = -AY$, if we set $Z = PY$ then $\partial_x Z = -AZ$. Setting

$$Z = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = P \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

we have

$$\partial_x \zeta_1 = \zeta_2,$$

$$\zeta_1 = ay + b\partial_x y.$$

Owing to $\mathfrak{L}(\zeta_1) = 0$ for all $y \in V_{\mathfrak{L}}$, $a + b\partial_x$ maps $V_{\mathfrak{L}}$ to $V_{\mathfrak{L}}$ and so $\mathfrak{R} := a + b\partial_x \in \mathcal{E}(\mathfrak{L})$.

2. If $\mathfrak{R} = a + b\partial_x \in \mathcal{E}(\mathfrak{L})$, then let $\zeta := \mathfrak{R}(y) = ay + b\partial_x y$. Then $\mathfrak{L}(\mathfrak{R}(y)) = \mathfrak{L}(\zeta) = 0$ and

$$\partial_x \begin{pmatrix} \zeta \\ \partial_x \zeta \end{pmatrix} = -A \begin{pmatrix} \zeta \\ \partial_x \zeta \end{pmatrix}, \quad \forall \begin{pmatrix} y \\ \partial_x y \end{pmatrix} \in \mathcal{E}(A).$$

Thus, there exists P satisfying

$$\begin{pmatrix} \zeta \\ \partial_x \zeta \end{pmatrix} = P \begin{pmatrix} y \\ \partial_x y \end{pmatrix},$$

which lead us to the expression

$$\zeta = ay + b\partial_x y,$$

$$\partial_x \zeta = (\partial_x a - bq)y + (a + b\partial_x - bp)\partial_x y.$$

Statements 3 and 4 can be obtained immediately. $\square$

Remark 1.1.26. Using this P, one can obtain the characteristic polynomial of P which is the characteristic polynomial of $\mathfrak{R}$ and so obtain eigenvalues of $\mathfrak{R}$ needed.

Remark 1.1.27. Let $\mathfrak{L}$ be the differential operator $\partial_x^2 + b$, where $b \in K$, $\mathcal{L} := \mathfrak{L}(y) = 0$. The dimension of the eigenring of $\mathfrak{L}$ is related with:

- the number of solutions over K of the differential equation $\mathcal{L}$ and its second symmetric power $\mathcal{L}^{\circledS 2}$ and
- the type of differential Galois group (see [7, 93, 43, 44, 45]).

The previous remark is detailed in the following lemma.

Lemma 1.1.28. *Assume $\mathfrak{L} = \partial_x^2 + b$, where $b \in K$, $\mathcal{L} := \mathfrak{L}(y) = 0$. The following statements holds.*

1. *If $\dim_{\mathcal{C}} \mathcal{E}(\mathfrak{L}) = 1$, then either differential Galois group is irreducible ($\mathbb{D}_\infty$ or primitive), or indecomposable ($G \subseteq \mathbb{B}$, $G \notin \{e, \mathbb{G}_m, \mathbb{G}_a, \mathbb{G}^{\{n\}}, \mathbb{G}^{[n]}\}$).*
2. *If $\dim_{\mathcal{C}} \mathcal{E}(\mathfrak{L}) = 2$, then either, the differential Galois group is the additive group or is contained in the multiplicative group, but never is the identity group. Thus, we can have two solutions but not over the differential field K.*
3. *If $\dim_{\mathcal{C}} \mathcal{E}(\mathfrak{L}) = 4$, then the differential Galois group is the identity group. In this case we have 2 independent solutions ζ_1 and ζ_2 in which ζ_1^2, ζ_2^2 and $\zeta_1\zeta_2$ are elements of the differential field K, i.e. the solutions of $\mathcal{L}^{\circledS 2}$ belongs to the differential field K.*

Proof. Assume that $G = \mathrm{DGal}(L/K)$, then G commutes with each element $P \in \mathcal{E}(\mathfrak{L})$, i.e., $GP = PG$. Computing the dimension of the set of matrices satisfying $GP = PG$ we obtain $\dim_{\mathcal{C}} \mathcal{E}(\mathfrak{L})$ for each $G \subseteq \mathrm{SL}(2, \mathbb{C})$. □

The eigenring for a differential operator $\mathfrak{L}$ has been implemented in Maple. The function is `eigenring`, the calling sequences are `eigenring(L,domain)` and `endomorphism-charpoly(L, R, domain)`, where $\mathfrak{L}$ is a differential operator, $\mathfrak{R}$ is the differential operator in the output of eigenring. The argument `domain` describes the differential algebra. If this argument is the list `[Dx,x]` then the differential operators are notated with the symbols `Dx` and `x`, where `Dx` is the operator ∂_x.

Example. Consider $K = \mathbb{C}(x)$, $\mathfrak{L} = \partial_x^2 - \frac{6}{x^2}$ and $[A]$ where A is given by

$$A = \begin{pmatrix} 0 & -1 \\ -6x^{-2} & 0 \end{pmatrix}.$$

The Eigenring of $[A]$ and the Eigenring of $\mathfrak{L}$ are given by

$$\mathcal{E}(A) = \mathrm{Vect}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & x \\ 6x^{-1} & 0 \end{pmatrix}, \begin{pmatrix} -3x^5 & x^6 \\ -9x^4 & 3x^5 \end{pmatrix}, \begin{pmatrix} 2x^{-5} & x^{-4} \\ -4x^{-6} & -2x^{-5} \end{pmatrix}\right),$$

$$\mathcal{E}(\mathfrak{L}) = \mathrm{Vect}\left(1, x\partial_x - 1, x^6\partial_x - 3x^5, \frac{\partial_x}{x^4} + \frac{2}{x^5}\right).$$

1.1.4 Riemann's Equation

The Riemann's equation is an important differential equation which has been studied for a long time, since Gauss, Riemann, Schwartz, etc., see for example [52, 76]. We are interested in the relationship with the Picard-Vessiot theory. Thus, we follow the works of Kimura [54], Martinet & Ramis [64] and Duval & Loday-Richaud [33].

Definition 1.1.29. The *Riemann's equation* is an homogeneous ordinary linear differential equation of the second order over the Riemann's sphere with at most three singularities which are of the regular type. Assuming a, b and c as regular singularities, the Riemann's equation may be written in the form

$$\begin{aligned}\partial_x^2 y + \left(\frac{1-\rho-\rho'}{x-a} + \frac{1-\sigma-\sigma'}{x-b} + \frac{1-\tau-\tau'}{x-c}\right)\partial_x y \\ + \left(\frac{\rho\rho'(a-b)(a-c)}{(x-a)^2(x-b)(x-c)} + \frac{\sigma\sigma'(b-a)(b-c)}{(x-b)^2(x-a)(x-c)} + \frac{\tau\tau'(c-a)(c-b)}{(x-c)^2(x-a)(x-b)}\right) y = 0,\end{aligned} \tag{1.12}$$

where (ρ, ρ'), (σ, σ') and (τ, τ') are the exponents at the singular points a, b, c respectively and must satisfy the Fuchs relation $\rho + \rho' + \sigma + \sigma' + \tau + \tau' = 1$. The quantities $\rho' - \rho$, $\sigma' - \sigma$ and $\tau' - \tau$ are called the *exponent differences* of the Riemann's equation (1.12) at a, b and c respectively and are denoted by $\widetilde{\lambda}$, $\widetilde{\mu}$ and $\widetilde{\nu}$ as follows:

$$\widetilde{\lambda} = \rho' - \rho, \quad \widetilde{\mu} = \sigma' - \sigma, \quad \widetilde{\nu} = \tau' - \tau.$$

The complete set of solutions of the Riemann's equation (1.12) is denoted by the symbol

$$y = P\left\{\begin{array}{cccc} a & b & c & \\ \rho & \sigma & \tau & x \\ \rho' & \sigma' & \tau' & \end{array}\right\}$$

and is called *Riemann's P-function.*

Now, we will briefly describe here the Kimura's theorem that gives necessary and sufficient conditions for the integrability of the Riemann's differential equation.

Theorem 1.1.30 (Kimura, [54])**.** *The Riemann's differential equation* (1.12) *is integrable if and only if, either*

(i) At least one of the four numbers $\widetilde{\lambda}+\widetilde{\mu}+\widetilde{\nu}$, $-\widetilde{\lambda}+\widetilde{\mu}+\widetilde{\nu}$, $\widetilde{\lambda}-\widetilde{\mu}+\widetilde{\nu}$, $\widetilde{\lambda}+\widetilde{\mu}-\widetilde{\nu}$ is an odd integer, or

(ii) The numbers $\widetilde{\lambda}$ or $-\widetilde{\lambda}$, $\widetilde{\mu}$ or $-\widetilde{\mu}$ and $\widetilde{\nu}$ or $-\widetilde{\nu}$ belong (in an arbitrary order) to some of the following fifteen families

1	$1/2+l$	$1/2+m$	*arbitrary complex number*	
2	$1/2+l$	$1/3+m$	$1/3+q$	
3	$2/3+l$	$1/3+m$	$1/3+q$	$l+m+q$ *even*
4	$1/2+l$	$1/3+m$	$1/4+q$	
5	$2/3+l$	$1/4+m$	$1/4+q$	$l+m+q$ *even*
6	$1/2+l$	$1/3+m$	$1/5+q$	
7	$2/5+l$	$1/3+m$	$1/3+q$	$l+m+q$ *even*
8	$2/3+l$	$1/5+m$	$1/5+q$	$l+m+q$ *even*
9	$1/2+l$	$2/5+m$	$1/5+q$	$l+m+q$ *even*
10	$3/5+l$	$1/3+m$	$1/5+q$	$l+m+q$ *even*
11	$2/5+l$	$2/5+m$	$2/5+q$	$l+m+q$ *even*
12	$2/3+l$	$1/3+m$	$1/5+q$	$l+m+q$ *even*
13	$4/5+l$	$1/5+m$	$1/5+q$	$l+m+q$ *even*
14	$1/2+l$	$2/5+m$	$1/3+q$	$l+m+q$ *even*
15	$3/5+l$	$2/5+m$	$1/3+q$	$l+m+q$ *even*

Here n, m, q are integers.

Using the Möebius transformation [41], also known as homographic substitution, in the Riemann's equation (1.12), we can map $x = a, b, c$ to $x' = a', b', c'$, respectively:

$$x' = \frac{px+q}{rx+s}.$$

In particular, we can place the singularities at $x = 0, 1, \infty$ to obtain the following Riemann's equation:

$$\begin{aligned}\partial_x^2 y + \left(\frac{1-\rho-\rho'}{x} + \frac{1-\sigma-\sigma'}{x-1}\right)\partial_x y \qquad\qquad\qquad (1.13)\\ + \left(\frac{\rho\rho'}{x^2} + \frac{\sigma\sigma'}{(x-1)^2} + \frac{\tau\tau'-\rho\rho'\sigma\sigma'}{x(x-1)}\right) y = 0,\end{aligned}$$

where the set of solutions is

$$y = P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ \rho & \sigma & \tau & x \\ \rho' & \sigma' & \tau' & \end{array}\right\}.$$

Sometimes it is very useful to map $x = 0, 1, \infty$ to $x' = -1, 1, \infty$ in the Riemann's equation (1.13), for example, setting $\rho = 0$, we can state the substitution:

$$P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & \sigma & \tau & x \\ \frac{1}{2} & \sigma' & \tau' & \end{array}\right\} = P\left\{\begin{array}{cccc} -1 & 1 & \infty & \\ \sigma & \sigma & 2\tau & \sqrt{x} \\ \sigma' & \sigma' & 2\tau' & \end{array}\right\}.$$

We can transforms equation (1.13) to the *Gauss Hypergeometric equation* as follows:

$$P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ \rho & \sigma & \tau & x \\ \rho' & \sigma' & \tau' & \end{array}\right\} = x^{\rho}(x-1)^{\sigma} P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & \kappa & x \\ 1-\gamma & \gamma-\kappa-\beta & \beta & \end{array}\right\},$$

where $\kappa = \rho+\sigma+\tau$, $\beta = \rho+\sigma+\tau'$ and $\gamma = 1+\rho-\rho'$. Then

$$y = P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & \kappa & x \\ 1-\gamma & \gamma-\kappa-\beta & \beta & \end{array}\right\},$$

is the set of solutions of the Gauss Hypergeometric differential equation[1]

$$\partial_x^2 y + \frac{(\gamma-(\kappa+\beta+1)x)}{x(1-x)}\partial_x y - \frac{\kappa\beta}{x(1-x)} y = 0, \tag{1.14}$$

where the Fuchs relation is trivially satisfied and the exponent differences are given by

$$\widetilde{\lambda} = 1-\gamma, \quad \widetilde{\mu} = 1-\gamma-\beta, \quad \widetilde{\nu} = \beta-\kappa.$$

We remark that the Galoisian structure of the Riemann's equation do not change with the Möebius transformation.

The *confluent Hypergeometric equation* is a degenerate form of the Hypergeometric differential equation where two of the three regular singularities merge into an irregular singularity. For example, making "1 tend to ∞" in a suitable way, the Hypergeometric equation (1.14) has two classical forms:

- *Kummer's* form
$$\partial_x^2 y + \frac{c-x}{x}\partial_x y - \frac{a}{x} y = 0 \tag{1.15}$$

- *Whittaker's* form
$$\partial_x^2 y = \left(\frac{1}{4} - \frac{\kappa}{x} + \frac{4\mu^2-1}{4x^2}\right) y \tag{1.16}$$

where the parameters of the two equations are linked by $\kappa = \frac{c}{2} - a$ and $\mu = \frac{c}{2} - \frac{1}{2}$. Furthermore, using the expression (1.3), we can see that the Whittaker's equation is the reduced form of the Kummer's equation. The Galoisian structure of these equations has been deeply studied in [64, 33].

Theorem 1.1.31 (Martinet & Ramis, [64]). *The Whittaker's differential equation* (1.16) *is integrable if and only if either,* $\kappa+\mu \in \frac{1}{2}+\mathbb{N}$, *or* $\kappa-\mu \in \frac{1}{2}+\mathbb{N}$, *or* $-\kappa+\mu \in \frac{1}{2}+\mathbb{N}$, *or* $-\kappa-\mu \in \frac{1}{2}+\mathbb{N}$.

[1] In general, for the Hypergeometric differential equation, is used α instead of κ, but we want to avoid further confusions.

The *Bessel's equation* is a particular case of the confluent Hypergeometric equation and is given by

$$\partial_x^2 y + \frac{1}{x}\partial_x y + \frac{x^2 - n^2}{x^2} y = 0. \tag{1.17}$$

Under a suitable transformation, the reduced form of the Bessel's equation is a particular case of the Whittaker's equation. Thus, we can obtain the following well known result, see [55, p. 417] and see also [56, 67].

Corollary 1.1.32. *The Bessel's differential equation* (1.17) *is integrable if and only if* $n \in \frac{1}{2} + \mathbb{Z}$.

We point out that the integrability of Bessel's equation for half integer of the parameter was known by Daniel Bernoulli [104]. By double confluence of the Hypergeometric equation (1.14), that is making "0 and 1 tend to ∞" in a suitable way, one gets the *parabolic cylinder equation* (also known as *Weber's* equation):

$$\partial_x^2 y = \left(\frac{1}{4}x^2 - \frac{1}{2} - n\right) y, \tag{1.18}$$

which is integrable if and only if $n \in \mathbb{Z}$, see [56, 33]. Setting $n = \frac{b^2 - c}{2a} - \frac{1}{2}$ and making the change $x \mapsto \sqrt{\frac{2}{a}}(ax + b)$, one can gets the Rehm's form of the Weber's equation:

$$\partial_x^2 y = \left(ax^2 + 2bx + c\right) y, \quad a \neq 0, \tag{1.19}$$

so that $\frac{b^2 - c}{a}$ is an odd integer.

The Hypergeometric equation, including confluences, is a particular case of the differential equation

$$\partial_x^2 y + \frac{L}{Q}\partial_x y + \frac{\lambda}{Q} y, \quad \lambda \in \mathbb{C}, \quad L = a_0 + a_1 x, \quad Q = b_0 + b_1 x + b_2 x^2. \tag{1.20}$$

We recall that the *classical orthogonal polynomials* and *Bessel polynomials* are solutions of equation (1.20), see [23, 51, 70]:

- Hermite, denoted by H_n,
- Chebyshev of first kind, denoted by T_n,
- Chebyshev of second kind, denoted by U_n,
- Legendre, denoted by P_n,
- Laguerre, denoted by L_n,
- associated Laguerre, denoted by $L_n^{(m)}$,
- Gegenbauer, denoted by $C_n^{(m)}$

- Jacobi polynomials, denoted by $\mathcal{P}_n^{(m,\nu)}$ and
- Bessel polynomials, denoted by B_n.

In the following table we give Q, L and λ corresponding to equation (1.20) for classical orthogonal polynomials and Bessel polynomials.

Polynomial	$\boldsymbol{Q}$	$\boldsymbol{L}$	$\boldsymbol{\lambda}$
H_n	1	$-2x$	$2n$
T_n	$1-x^2$	$-x$	n^2
U_n	$1-x^2$	$-3x$	$n(n+2)$
P_n	$1-x^2$	$-2x$	$n(n+1)$
L_n	x	$1-x$	n
$L_n^{(m)}$	x	$m+1-x$	n
$C_n^{(m)}$	$1-x^2$	$-(2m+1)x$	$n(n+2m)$
$\mathcal{P}_n^{(m,\nu)}$	$1-x^2$	$\nu-m-(m+\nu+2)x$	$n(n+1+m+\nu)$
B_n	x^2	$2(x+1)$	$-n(n+1)$

The *associated Legendre polynomials*, denoted by $P_n^{(m)}$, does not appear in the previous table. They are solutions of the differential equation

$$\partial_x^2 y - \frac{2x}{1-x^2}\partial_x y + \left(\frac{n(n+1)-\frac{m^2}{1-x^2}}{1-x^2}\right) y = 0. \tag{1.21}$$

This equation can be transformed into a Riemann's differential equation through the change $x \mapsto \frac{1}{1-x^2}$. Thus, the complete set of solutions of equation (1.21) is given by

$$P\left\{ \begin{array}{cccc} 0 & 1 & \infty & \\ -\frac{1}{2}n & 0 & \frac{1}{2}m & \frac{1}{1-x^2} \\ \frac{1}{2}+\frac{1}{2}n & \frac{1}{2} & -\frac{1}{2}m & \end{array} \right\},$$

the exponent differences are $\widetilde{\lambda} = \frac{1}{2}$, $\widetilde{\mu} = \frac{1}{2}$ and $\widetilde{\nu} = 0$. By Kimura's theorem this equation is integrable.

Finally, we remark that integrability conditions and solutions of differential equations with solutions orthogonal polynomials, including Bessel polynomials, can be obtained applying Kovacic's algorithm. In the same way, we can apply Kovacic's algorithm to obtain the same results given by Kimura [54] and Martinet & Ramis [64]. Also we recall that Duval & Loday-Richaud applied Kovacic's algorithm to some families of special functions [33].

1.2 Supersymmetric Quantum Mechanics

In this section we establish the basic information on Supersymmetric Quantum Mechanics. We only consider the case of non-relativistic quantum mechanics.

1.2.1 The Schrödinger Equation

In classical mechanics the Hamiltonian corresponding to the energy (kinetic plus potential) is given by

$$H = \frac{\|\overrightarrow{p}\|^2}{2m} + U(\overrightarrow{x}), \quad \overrightarrow{p} = (p_1, \dots, p_n), \quad \overrightarrow{x} = (x_1, \dots, x_n),$$

while in quantum mechanics the momentum $\overrightarrow{p}$ is given by $\overrightarrow{p} = -\imath\hbar\nabla$, the Hamiltonian operator is the Schrödinger (non-relativistic, stationary) operator which is given by

$$H = -\frac{\hbar^2}{2m}\nabla^2 + U(\overrightarrow{x})$$

and the Schrödinger equation is $H\Psi = E\Psi$, where $\overrightarrow{x}$ is the *coordinate*, the eigenfunction Ψ is the *wave function*, the eigenvalue E is the *energy level*, $U(\overrightarrow{x})$ is the *potential or potential energy* and the solutions of the Schrödinger equation are the *states* of the particle. Furthermore, it is known that $H^\dagger = H$, i.e., the Schrödinger operator is a *self-adjoint operator* in a suitable *complex and separable Hilbert space*. Thus, H has a purely real *spectrum* $\mathrm{spec}(H)$ and its spectrum $\mathrm{spec}(H)$ is the disjoint union of the *point spectrum* $\mathrm{spec}_p(H)$ and the continuous spectrum $\mathrm{spec}_c(H)$, i.e., $\mathrm{spec}(H) = \mathrm{spec}_p(H) \cup \mathrm{spec}_c(H)$ with $\mathrm{spec}_p(H) \cap \mathrm{spec}_c(H) = \emptyset$. See for example [6, 78, 98].

Along this thesis we only consider the one-dimensional Schrödinger equation written as follows:

$$H\Psi = E\Psi, \quad H = -\partial_z^2 + V(z), \tag{1.22}$$

where $z = x$ (cartesian coordinate) or $z = r$ (radial coordinate) and $\hbar = 2m = 1$. We denote by Ψ_n the wave function for $E = E_n$. The potentials should satisfy some conditions depending of the physic situation such as barrier, scattering, etc., see [24, 36, 60, 66, 88].

Definition 1.2.1 (Bound States)**.** The solution Ψ_n is called a *bound state* when E belongs to the point spectrum of H and its norm is finite, i.e.,

$$E_n \in \mathrm{spec}_p(H), \quad \int |\Psi_n(x)|^2 dx < \infty, \quad n \in \mathbb{Z}_+. \tag{1.23}$$

An interesting property of bound states is given by the Sturm's theorem, see [6, 98].

Theorem 1.2.2 (Sturm's Theorem). *If $\Psi_0, \Psi_1, \ldots, \Psi_n, \ldots$ are the wave functions of the bound states with energies $E_0 < E_1 < \cdots < E_n < \ldots$, then Ψ_n has n nodes (zeros). Furthermore, between two consecutive nodes of Ψ_n, there is a node of Ψ_{n-1}, and moreover Ψ_{n+r} has at least one zero for all $r \geq 1$.*

Definition 1.2.3 (Ground State and Excited States). Assume $\Psi_0, \Psi_1, \ldots, \Psi_n, \ldots$ as in the Sturm's theorem. The state Ψ_0, which is state with minimum energy is called the *ground state* and the states $\Psi_1, \ldots, \Psi_n, \ldots$ are called the *excited states*.

Definition 1.2.4 (Scattering States). The solution Ψ corresponding to the level energy E is called a *scattering state* when E belongs to the continuous spectrum of H and its norm is infinite.

The wave function belonging to the continuous spectrum have two typical boundary conditions: the first ones, barrier potentials and the second one periodic boundary conditions. The *transmission and reflection coefficients* are related with the barrier potentials, [36].

When the particle moves in one dimension, we use the classical one dimensional Schrödinger equation with cartesian coordinate x.

Example (The Harmonic Oscillator). We consider the Hamiltonian operator H given by (1.22). Normalizing the angular velocity ($\omega = 1$), the one-dimensional harmonic oscillator potential is $V = \frac{1}{4}x^2$. The *creator* (raising) and *annihilator* (lowering) operators, denoted respectively by $a^\dagger$ and a, given by

$$a^\dagger = -\partial_x + \frac{1}{2}x, \quad a = \partial_x + \frac{1}{2}x$$

lead us to the relations

$$H = a^\dagger a + \frac{1}{2}, \quad [a, a^\dagger] = 1, \quad [a^\dagger, H] = -a^\dagger, \quad [a, H] = a. \tag{1.24}$$

We want to solve $H\Psi = E\Psi$, in particular we are interested in the case $H\Psi_n = E_n\Psi_n$, where Ψ_0 is the ground state and $\Psi_1, \ldots, \Psi_n$ are the excited states. Considering $H\Psi_0 = E_0\Psi_0$ and assuming that $a^\dagger a\Psi_0 = 0$, we obtain the energy E_0 and the ground state Ψ_0:

$$E_0 = \frac{1}{2}, \quad \Psi_0 = e^{-\frac{1}{4}x^2}.$$

Using the relation $\Psi_n = (a^\dagger)^n\Psi_0$, we obtain the rest of wave functions

$$\Psi_0 = e^{-\frac{1}{4}x^2}, \qquad \Psi_1 = a^\dagger\Psi_0 = xe^{-\frac{1}{4}x^2}, \qquad \Psi_2 = a^\dagger\Psi_1 = (x^2-1)e^{-\frac{1}{4}x^2},$$

$$\Psi_3 = (x^3 - 3x)\Psi_0, \quad \Psi_4 = (x^4 - 6x^2 + 3)\Psi_0, \ldots \quad \Psi_n = a^\dagger\Psi_{n-1} = H_n\Psi_0,$$

where H_n are the Hermite's polynomials presented in the previous section. Now, to obtain the complete energy spectrum we use the relations (1.24), thus

$$E_0 = \frac{1}{2}, \quad E_1 = 1 + E_0 = \frac{3}{2}, \quad E_2 = 1 + E_1 = \frac{5}{2}, \quad \ldots, \quad E_n = \frac{2n+1}{2}.$$

Another way to obtain E_n is given by the formula

$$E_n = \frac{1}{4}x^2 - \frac{\partial_x^2 \Psi_n}{\Psi_n}.$$

Also we can consider the particle moving in three dimensions, this means that $\overrightarrow{p} = (p_x, p_y, p_z)$, where $p_x = -\imath\partial_x$, $p_y = -\imath\partial_y$, $p_z = -\imath\partial_z$ and $p = \|\overrightarrow{p}\|$. The *angular momentum operator* is given by $\overrightarrow{L} = (L_x, L_y, L_z)$ where $L_x = yp_z - zp_y$, $L_y = zp_x - xp_z$, $L_z = xp_y - yp_x$ and $L = \|\overrightarrow{L}\|$. The square of the angular momentum operator $\|\overrightarrow{L}\|^2 = L^2 = L_x^2 + L_y^2 + L_z^2$ commutes with all components of the angular momentum operator.

In spherical coordinates $x = r\sin\nu\cos\varphi$, $y = r\sin\nu\sin\varphi$, $z = r\cos\nu$, L^2 is given by

$$L^2 = -\triangle_{\nu,\varphi}, \quad \triangle_{\nu,\varphi} = \frac{1}{\sin\nu}\partial_\nu\left(\sin\nu\partial_\nu\right) + \frac{1}{\sin^2\nu}\partial_\varphi^2,$$

where we denote by $\triangle_{\nu,\varphi}$ that part of the Laplacian acting on the variables ν and φ only. The kinetic energy given by $p^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ reads in polar coordinates as

$$p^2 = p_r^2 + \frac{1}{r^2}L^2, \quad p_r^2 = -\frac{1}{r^2}\partial_r\left(r^2\partial_r\right) = -\left(\partial_r^2 + \frac{2}{r}\partial_r\right).$$

Now, for *central potentials*, where the potential $U(\overrightarrow{r})$ is spherically symmetric, i.e., $U(\overrightarrow{r}) = U(r)$, we can reduce the Schrödinger equation to an one dimensional problem, the so-called *radial equation*.

We start writing the eigenfunctions and eigenvalues of the operator L^2:

$$L^2 Y_{\ell,m}(\nu,\varphi) = \ell(\ell+1)Y_{\ell,m}(\nu,\varphi),$$

the eigenfunctions $Y(\nu,\varphi)$ are the *spherics harmonics* which are related with the associated Legendre Polynomials

$$Y_{\ell,m}(\nu,\varphi) = P_\ell^m(\cos\nu)e^{\imath m\varphi}.$$

Assuming Φ as eigenfunctions of $p^2 + U(r)$ satisfying $\Phi = R_\ell(r)Y_{\ell,m}(\nu,\varphi)$, i.e., the *partial wave function decomposition* see ([36, 60, 88]), we have

$$\left(p_r^2 + \frac{1}{r^2}L^2 + U(r) - E\right)R_\ell(r)Y_{\ell,m}(\nu,\varphi) = 0,$$

so that we obtain

$$\left(p_r^2 + U(r) - E\right) R_\ell(r) + \frac{R_\ell(r)}{r^2 Y_{\ell,m}(\nu,\varphi)} L^2 Y_{\ell,m}(\nu,\varphi) = 0$$

and owing to $L^2 Y_{\ell,m}(\nu,\varphi) = \ell(\ell+1) Y_{\ell,m}(\nu,\varphi)$ we have the radial equation

$$\left(p_r^2 + \frac{\ell(\ell+1)}{r^2} + U(r)\right) R_\ell(r) = E R_\ell(r).$$

Applying the expression (1.3), the radial equation can be reduced to the Schrödinger equation (1.22) as follows:

$$H\Psi = E\Psi, \quad H = -\partial_r^2 + V(r), \quad V(r) = \frac{\ell(\ell+1)}{r^2} + U(r), \quad \Psi = rR_\ell(r).$$

The equation for the angular part is always solved through spherics harmonics, while for the radial part, the analysis depends on the spherically symmetric potential $U(r)$. One example of the radial equation is the Coulomb potential. The complete set of physical and mathematical conditions for the potentials, spectrum and wave functions in one or three dimensions can be found in any book of quantum mechanics, including bound states and scattering cases, see for example [36, 60, 88].

1.2.2 Darboux Transformation

The following theorem is the most general case for Darboux transformation in the case of second order linear differential equations, which is taken faithfully from [28].

Theorem 1.2.5 (Darboux, [28]). *Suppose that we know how to integrate, for any value of the constant m, the following equation*

$$\partial_x^2 y + P\partial_x y + (Q - mR)y = 0. \tag{1.25}$$

If $\theta \neq 0$ is an integral of the equation

$$\partial_x^2 \theta + P\partial_x \theta + Q\theta = 0,$$

then the function

$$u = \frac{\partial_x y - \frac{\partial_x \theta}{\theta} y}{\sqrt{R}}, \tag{1.26}$$

will be an integral of the equation

$$\partial_x^2 u + P\partial_x u + \left(\theta\sqrt{R}\partial_x\left(\frac{P}{\theta\sqrt{R}}\right) - \theta\sqrt{R}\partial_x^2\left(\frac{1}{\theta\sqrt{R}}\right) - mR\right)u = 0, \tag{1.27}$$

for $m \neq 0$.

Darboux in [28, 29] presented the particular case for $R = 1$ and $P = 0$, which today is known as *Darboux transformation*, but really is a corollary of the *general Darboux transformation* given in theorem 1.2.5.

Corollary 1.2.6 (Darboux, [28, 29])**.** *Suppose that we know how to integrate*

$$\partial_x^2 y = (f(x) + m)y \tag{1.28}$$

for any value of m. If θ satisfies the equation $\partial_x^2\theta = (f(x) + m_1)\theta$, the function

$$u = \partial_x y - \frac{\partial_x \theta}{\theta} y$$

will be an integral of the equation

$$\partial_x^2 u = \left(\theta \partial_x^2 \left(\frac{1}{\theta}\right) - m_1 + m\right) u, \tag{1.29}$$

for $m \neq m_1$. Furthermore,

$$\theta \partial_x^2 \left(\frac{1}{\theta}\right) - m_1 = f(x) - 2\partial_x \left(\frac{\partial_x \theta}{\theta}\right) = 2\left(\frac{\partial_x \theta}{\theta}\right)^2 - f(x) - 2m_1.$$

Remark 1.2.7. In practice, we need two values of m to apply the Darboux's results.

Example. Consider the equation $\partial_x^2 y = my$. Employing the solution $\theta = x$, we shall get

$$\partial_x^2 y = \left(\frac{1 \cdot 2}{x^2} + m\right) y.$$

Applying the same method to the latter equation, but taking now $\theta = x^2$, we shall have

$$\partial_x^2 y = \left(\frac{2 \cdot 3}{x^2} + m\right) y$$

and so on. The cases $m_1 = 0$ and $m_1 = -1$ also can be found as exercises in the Ince's book [49, p. 132].

We can see that equation (1.28) coincides with the Schrödinger equation (1.22). Thus, we can apply the Darboux transformation in where $m = -E$ and $m_1 = -E_0$.

The following definition corresponds with Delsarte's transformation operators (isomorphisms of transmutations), which today are called *intertwiner operators*, see [30].

Definition 1.2.8. Two operators $\mathfrak{L}_0$ and $\mathfrak{L}_1$ are said to be *intertwined* by an operator T if

$$\mathfrak{L}_1 T = T \mathfrak{L}_0. \tag{1.30}$$

We can relate the intertwiner operators with the Darboux transformation of equation (1.22), where $\mathfrak{L}_1$ and $\mathfrak{L}_0$ are Schrödinger operators and T can be either $\mp\partial_x + \frac{\partial_x \Psi_0}{\Psi_0}$.

Crum, inspired by the works of Liouville [61, 62] obtained one kind of iterative generalization of Darboux's result giving emphasis in the Sturm-Liouville systems, i.e., he proved that the Sturm-Liouville conditions are preserved under Darboux transformations, see [27]. The Crum's result is presented in the following theorem, defining the Wronskian determinant W of k functions $f_1, f_2, \ldots, f_k$ by

$$W(f_1, \ldots, f_k) = \det A, \quad A_{ij} = \partial_x^{i-1} f_j, \quad i, j = 1, 2, \ldots, k.$$

Theorem 1.2.9 (Crum, [27]). *Let $\Psi_1, \Psi_2, \ldots \Psi_n$ be solutions of the Schrödinger equation* (1.22) *for fixed, arbitrary energy levels $E = E_1, E_2, \ldots, E_n$, respectively. Then, we obtain the Schrödinger equation*

$$H^{[n]}\Psi[n] = E\Psi[n], \quad E \neq E_i, 1 \leq i \leq n, \quad H^{[n]} = -\partial_x^2 + V[n],$$

where

$$\Psi[n] = \frac{W(\Psi_1, \ldots, \Psi_n, \Psi)}{W(\Psi_1, \ldots, \Psi_n)}, \quad V[n] = V - 2\partial_x^2 \ln W(\Psi_1, \ldots, \Psi_n).$$

Darboux transformation coincides with Crum's result in the case $n = 1$ and the iterations of Darboux transformation coincides with Crum iteration, see [71]. Both formalisms allow us to obtain new families of Schrödinger equations preserving the spectrum and the Sturm-Liouville conditions, see [66, 71]. Furthermore, there are extensions of Crum's iteration connecting the Sturm-Liouville theory with orthogonal polynomial theory [58].

Schrödinger in [89] factorized the Hypergeometric equation (1.14). He started making the change of variable $2x - 1 = \cos\theta$, after, using the expression (1.3), he reduced the Hypergeometric equation to obtain conditions of factorization. In this way, setting $\kappa\beta = E$, we obtain families of Schrödinger equations (1.22). This result was used by Natanzon in [69] to obtain the well known *Natanzon's potentials*, i.e, potentials which can be obtained by transformations of the Hypergeometric equation and its confluences, see [24, 25]. In particular, the *Ginocchio potentials* are obtained through the Gegenbauer polynomials.

Witten in [107, §6] presented some models in where *dynamical breaking of supersymmetry* is plausible. The first model is a model in potential theory-supersymmetric quantum mechanics, although is not a model in the field theory.

Definition 1.2.10. A *supersymmetric quantum mechanical system* is one in which there are operators Q_i, that commute with the hamiltonian $\mathcal{H}$,

$$[Q_i, \mathcal{H}] = 0, \quad i = 1, \ldots, n \tag{1.31}$$

and satisfy the algebra

$$\{Q_i, Q_j\} = \delta_{ij}\mathcal{H}, \quad \{Q_i, Q_j\} = Q_iQ_j + Q_jQ_i. \tag{1.32}$$

The simplest example of a supersymmetric quantum mechanical system corresponds to the case $n = 2$, which in physical sense involves a *spin one half particle* moving on the line. This case is the main object of this thesis. The wave function of $\mathcal{H}\Phi = E\Phi$ is therefore a two-component *Pauli spinor*,

$$\Phi(x) = \begin{pmatrix} \Psi_+(x) \\ \Psi_-(x) \end{pmatrix}.$$

The *supercharges* Q_i are defined as

$$Q_\pm = \frac{\sigma_1 p \pm \sigma_2 W(x)}{2}, \quad Q_+ = Q_1,\, Q_- = Q_2, \quad p = -i\partial_x, \tag{1.33}$$

where the *superpotential* W is an arbitrary function of x and σ_i are the usual *Pauli spin matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using the expressions (1.31), (1.32) and (1.33) we obtain $\mathcal{H}$:

$$\mathcal{H} = 2Q_-^2 = 2Q_+^2 = \frac{I_2 p^2 + I_2 W^2(x) + \sigma_3 \partial_x W(x)}{2}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.34}$$

The *supersymmetric partner Hamiltonians* $H_\pm$ are given by

$$H_\pm = -\frac{1}{2}\partial_x^2 + V_\pm, \quad V_\pm = \left(\frac{W}{\sqrt{2}}\right)^2 \pm \frac{1}{\sqrt{2}}\partial_x \left(\frac{W}{\sqrt{2}}\right).$$

The potentials $V_\pm$ are called *supersymmetric partner potentials* and are linked with the superpotential W through a Riccati equation. So that equation (1.34) can be written as

$$\mathcal{H} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix},$$

which lead us to the Schrödinger equations $H_+\Psi_+ = E\Psi_+$ and $H_-\Psi_- = E\Psi_-$, and for instance, to solve $\mathcal{H}\Phi = E\Phi$ is equivalent to solve simultaneously $H_+\Psi_+ = E\Psi_+$ and $H_-\Psi_- = E\Psi_-$.

We analyze equation $Q_i\Phi = 0$, which must be satisfied by a *supersymmetric state*. This is due to $Q_1^2 = Q_2^2 = \frac{1}{2}\mathcal{H}$ and for instance $Q_2 = -i\sigma_3 Q_1$, which implies that $Q_1\Phi = 0$, or $\sigma_1 p\Phi = -\sigma_2 W\Phi$. Now, multiplying by σ_1 and using the facts that $p = -i\partial_x$, $\sigma_1\sigma_2 = i\sigma_3$, this equation becomes

$$\partial_x \Phi = W(x)\sigma_3\Phi(x), \tag{1.35}$$

and the solution is

$$\Phi(x) = e^{\int W(x)\sigma_3 dx}. \tag{1.36}$$

In agreement with [107], one important generalization of the previous model could be its extension to four dimensions, i.e., the case $n = 4$.

V.B. Matveev and M. Salle in [66] interpret the Darboux Theorem as Darboux covariance of a Sturm-Liouville problem and proved the following result, see also [86, §5-6].

Theorem 1.2.11 (Matveev & Salle, [66]). *The case $n = 2$ in Supersymetric Quantum Mechanics is equivalent to a single Darboux transformation.*

According to Natanzon [69], a *solvable potential*, also known as *exactly solvable potentials*, is a potential in which the Schrödinger equation can be reduced to hypergeometric or confluent hypergeometric form. The following are examples of solvable potentials.

$$\begin{array}{ll}
\textbf{Potential} & \textbf{Name} \\
V(x) = \begin{cases} 0, & x \in [0, L] \\ \infty, & x \notin [0, L] \end{cases} & \textit{Infinite square well} \\
V(r) = \begin{cases} \frac{\ell(\ell+1)}{r^2}, & r \in [0, L] \\ \infty, & r \notin [0, L] \end{cases} & \textit{Radial infinite square well} \\
V(r) = -\frac{\mu}{e^{\kappa r}-1}, & \textit{Hulthén} \\
V(r) = -\frac{\mu}{e^{\kappa r}-1} + \frac{2\kappa^2 e^{\kappa r}}{(e^{\kappa r}-1)^2}, & \textit{Generalized Hulthén} \\
V(x) = \frac{1}{4}\omega^2 x^2 + g_a \frac{x^2-a^2}{(x^2+a^2)^2}, \quad g_a > 0 & ------ [22]
\end{array} \tag{1.37}$$

We remark that our definition of integrability, definition 1.1.8, is different of the concept of solvability given by Natanzon. In the next chapter we come back on this problem.

According to Dutt et al. [32], a *conditionally solvable potential* is a potential in which the entire bound state spectrum can be analytically obtained, where the parameters in the potential satisfies a specific relation. The following potentials are two examples of conditionally solvable potentials

$$V(x) = \frac{A}{1+e^{-2x}} - \frac{B}{\sqrt{1+e^{-2x}}} - \frac{3}{4(1+e^{-2x})^2} \quad \text{and}$$

$$V(x) = \frac{A}{1+e^{-2x}} - \frac{Be^{-x}}{\sqrt{e^{-2x}+1}} - \frac{3}{4(1+e^{-2x})^2}. \tag{1.38}$$

As a generalization of the method to solve the harmonic oscillator [37, 31], the ladder (raising and lowering) operators are defined as

$$A^+ = -\partial_x - \frac{\partial_x \Psi_0}{\Psi_0}, \quad A = \partial_x - \frac{\partial_x \Psi_0}{\Psi_0},$$

which are very closed with the supercharges $Q_\pm$ in the Witten's formalism. Thus,

$$A\Psi_0 = 0, \quad A^+ A = H_-, \quad AA^+ = H_+ = -\partial_x^2 + V_+(x), \quad \text{where}$$

$$V_+(x) = V_-(x) - 2\partial_x \left(\frac{\partial_x \Psi_0}{\Psi_0} \right) = -V_-(x) + 2 \left(\frac{\partial_x \Psi_0}{\Psi_0} \right)^2.$$

The supersymmetric partner potentials V_+ and V_- have the same energy levels, except for $E_0^{(-)} = 0$. In terms of the superpotential $W(x)$, the operators A and A^+ are given by

$$A^+ = -\partial_x + W(x), \quad A = \partial_x + W(x).$$

In the same way, the supersymmetric partner potentials $V_\pm(x)$ and the superpotential $W(x)$ satisfies:

$$\frac{V_+(x) + V_-(x)}{2} = W^2(x), \quad [A, A^+] = 2\partial_x W(x).$$

Let $\Psi_n^{(-)}$ and $\Psi_n^{(+)}$ denote the eigenfunctions of the supersymmetric Hamiltonians H_- and H_+ respectively, with eigenvalues $E_n^{(-)}$ and $E_n^{(+)}$. The integer $n = 0, 1, 2, \ldots,$ denotes the number of nodes in the wave function.

Theorem 1.2.12 (Dutt et al., [31]). *If $\Psi_n^{(-)}$ is any eigenfunction of H_- with eigenvalue $E_n^{(-)}$, then $A\Psi_n^{(-)}$ is an eigenfunction of H_+ with the same eigenvalue. Furthermore*

$$E_n^{(+)} = E_{n+1}^{(-)}, \quad \Psi_n^{(+)} = \frac{A}{\sqrt{E_{n+1}^{(-)}}} \Psi_{n+1}^{(-)}.$$

Considering $E_0 = 0$ and $n > 0$ in theorem 1.2.12, we can see that the supersymmetric partner potentials V_+ and V_- have the same spectrum. The ground state energy $E_0 = 0$ of V_- has no corresponding level for V_+. Furthermore, if the eigenfunction $\Psi_{n+1}^{(-)}$ of H_- is normalized, then also the eigenfunction $\Psi_n^{(+)}$ of H_+ is normalized.

The operator A converts an eigenfunction of H_- into an eigenfunction of H_+ with the same energy, whilst the operator A^+ converts an eigenfunction of H_+ into an eigenfunction of H_- with the same energy. Furthermore, the operator A destroys a node ($\Psi_{n+1}^{(-)}$ has $n+1$ nodes, whilst $\Psi_n^{(+)}$ has n nodes) and the operator A^+ creates a node. In summary, the *annihilator and creator operators* (A and A^+ respectively) connect states of the same energy for two different supersymmetric

partner potentials.

Gendenshteïn, in his remarkable paper [37], introduced the concept of *shape invariance*, which is a property or condition of some classes of potentials with respect to their parameters. Assuming a as a family of parameters, the shape invariance condition can be seen such as follows:

$$V_{n+1}(x;a_n) = V_n(x;a_{n+1}) + R(a_n), \quad V_- = V_0, \quad V_+ = V_1,$$

where R is a remainder, which does not depends on x. In this way we say that $V = V_- = V_0$ is a *shape invariant potential*.

The potentials, corresponding to a Schrödinger operator, satisfying the shape invariance property allow us to obtain a fully algebraic scheme for the spectrum and wave functions. This is illustrated in the following theorem obtained by Gendenshteïn.

Theorem 1.2.13 (Gendenshteïn, [37])**.** *Consider the Schrödinger equation $H\Psi = E_n\Psi$, where $V = V_- = V_0$ is a shape invariant potential. If we fix the first level of energy $E_0 = 0$, then the excited spectrum and the wave functions are given respectively by*

$$E_n = \sum_{k=2}^{n+1} R(a_k), \quad \Psi_n^{(-)}(x;a_1) = \prod_{k=1}^{n} A^+(x;a_k)\Psi_0^{(-)}(x;a_{n+1}). \tag{1.39}$$

The complete statement and proof of theorem 1.2.13 can be found in [37, 31]. As consequences we have the following facts (see [37, 31]):

- $$\Psi_n^{(+)}(x;a_0) = \Psi_n^{(-)}(x;a_1), \quad \Psi_{n+1}^{(-)}(x;a_0) = A^\dagger(x;a_0)\Psi_n^{(-)}(x;a_1).$$

- $$H^{(n)} = -\partial_x^2 + V_-(x;a_n) + \sum_{k=1}^{n} R(a_k)$$

  has the same spectrum of $H^{(n+1)}$ for $n > 0$, where

  $$H^{(0)} = H_-, \quad H^{(1)} = H_+, \quad a_n = f^n(a_0).$$

- $$H^{(n+1)} = -\partial_x^2 + V_+(x;a_n) + \sum_{k=1}^{n} R(a_k).$$

Following [24, 31] we present the list of shape invariant potentials given in expression (1.40).

Potential V	**Name**
$\frac{1}{4}\omega^2 x^2 - \frac{\omega}{2}$	*Harmonic Oscillator*
$\frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+1)}{r^2} - \left(\ell + \frac{3}{2}\right)\omega$	*3D Harmonic Oscillator*
$-\frac{e^2}{r} + \frac{\ell(\ell+1)}{r^2} + \frac{e^4}{8(\ell+1)^2}$	*Coulomb*
$A^2 + B^2 e^{-2ax} - 2B\left(A + \frac{a}{2}\right)e^{-ax}$	*Morse*
$A^2 + \frac{B^2}{A^2} - 2B\coth ar + A\frac{A-a}{\sinh^2 ar}$	*Eckart*
$A^2 + \frac{B^2}{A^2} + 2B\tanh ax - A\frac{A+a}{\cosh^2 ax}$	*Rosen-Morse Hyp.*
$-A^2 + \frac{B^2}{A^2} + 2B\cot ax + A\frac{A+a}{\sin^2 ar}$	*Rosen-Morse Trig.*
$A^2 + \frac{B^2-A^2-Aa}{\cosh^2 ax} + \frac{B(2A+a)\sinh ax}{\cosh^2 ax}$	*Scarf Hyp. I*
$A^2 + \frac{B^2+A^2+Aa}{\sinh^2 ar} - \frac{B(2A+a)\cosh ar}{\sinh^2 ar}$	*Scarf Hyp. II*
$-A^2 + \frac{B^2+A^2-Aa}{\cos^2 ax} - \frac{B(2A-a)\sin ax}{\cos^2 ax}$	*Scarf Trig. I*
$-A^2 + \frac{B^2+A^2-Aa}{\sin^2 ax} - \frac{B(2A-a)\cos ax}{\sin^2 ax}$	*Scarf Trig. II*
$-(A+B)^2 + \frac{A(A-a)}{\cos^2 ax} + \frac{B(B-a)}{\sin^2 ax}$	*Pöschl-Teller 1*
$(A-B)^2 - \frac{A(A+a)}{\cosh^2 ar} + \frac{B(B-a)}{\sinh^2 ar}$	*Pöschl-Teller 2*

(1.40)

We recall that a good short survey about Darboux transformations can be found in [86].

Chapter 2

Differential Galois Theory Approach to Supersymmetric Quantum Mechanics

In this chapter we present our original results of this thesis, which corresponds to the Galoisian approach to Supersymmetric quantum mechanics. We start rewriting in a Galoisian context some points of the section 1.2, chapter 1. The results presented here are also true for any differential field K with field of constants $\mathcal{C}$ in agreement with definition 1.1.4. We emphasize in the differential fields $K_0 = \mathbb{C}(x)$ and $K_1 = \mathbb{C}(z(x), \partial_x z(x))$, where in both cases $\mathcal{C} = \mathbb{C}$.

2.1 Preliminaries

The main object of our Galoisian analysis is the Schrödinger equation (1.22), which now is written as

$$\mathcal{L}_\lambda := H\Psi = \lambda\Psi, \quad H = -\partial_x^2 + V(x), \quad V \in K, \tag{2.1}$$

where K is a differential field (with $\mathbb{C}$ as field of constants). We are interested in the integrability of equation (2.1) in agreement with definition 1.1.8.

We introduce the following notations.

- Denote by $\Lambda \subseteq \mathbb{C}$ the set of eigenvalues λ such that equation (2.1) is integrable according with definition 1.1.8.
- Denote by Λ_+ the set $\{\lambda \in \Lambda \cap \mathbb{R} : \lambda \geq 0\}$ and by Λ_- the set $\{\lambda \in \Lambda \cap \mathbb{R} : \lambda \leq 0\}$.

- Denote by L_λ the Picard-Vessiot extension of $\mathcal{L}_\lambda$. Thus, the differential Galois group of $\mathcal{L}_\lambda$ is denoted by $\mathrm{DGal}(L_\lambda/K)$.

The set Λ will be called *the algebraic spectrum* (or alternatively *the Liouvillian spectral set*) of H. We remark that Λ can be $\emptyset$, i.e., $\mathrm{DGal}(L_\lambda/K) = \mathrm{SL}(2,\mathbb{C})$ $\forall\lambda \in \mathbb{C}$. On the other hand, by theorem 1.1.9, if $\lambda_0 \in \Lambda$ then $(\mathrm{DGal}(L_{\lambda_0}/K))^0 \subseteq \mathbb{B}$.

Definition 2.1.1 (Algebraically Solvable and Quasi-Solvable Potentials)**.** We say that the potential $V(x) \in K$ is:

- an *algebraically solvable potential* when Λ is an infinite set, or
- an *algebraically quasi-solvable potential* when Λ is a non-empty finite set, or
- an *algebraically non-solvable potential* when $\Lambda = \emptyset$.

When $\mathrm{Card}(\Lambda) = 1$, we say that $V(x) \in K$ is a *trivial* algebraically quasi-solvable potential.

Examples. Assume $K = \mathbb{C}(x)$.

1. If $V(x) = x$, then $\Lambda = \emptyset$, $V(x)$ is algebraically non-solvable, see [53, 56].
2. If $V(x) = 0$, then $\Lambda = \mathbb{C}$, i.e., $V(x)$ is algebraically solvable. Furthermore,
$$\mathrm{DGal}(L_0/K) = e, \quad \mathrm{DGal}(L_\lambda/K) = \mathbb{G}_m, \quad \lambda \neq 0.$$
3. If $V(x) = \frac{x^2}{4} + \frac{1}{2}$, then $\Lambda = \{n : n \in \mathbb{Z}\}$, $V(x)$ is algebraically solvable. This example corresponds to the Weber's equation, see subsection 1.1.4.
4. If $V(x) = x^4 - 2x$, then $\Lambda = \{0\}$, $V(x)$ is algebraically quasi-solvable (trivial).

Remark 2.1.2. We can obtain algebraically solvable and quasi-solvable potentials in the following ways.

- Giving the potential V, we try to solve the differential equation $\partial_x^2\Psi_0 = V\Psi_0$ expecting to obtain the superpotential $W = \partial_x \ln(\Psi_0)$. If the superpotential exists in the Liouvillian class (the differential equation is integrable), then we search Λ into the Schrödinger equation $H\Psi = \lambda\Psi$. With this method the algebraic spectrum can be the empty set. We will illustrate this in section 2.2 and 2.3.
- Giving the superpotential W, we construct the potential $V = \partial_x W + W^2$. After we can search the algebraic spectrum Λ in $H\Psi = \lambda\Psi$. With this method we have that $0 \in \Lambda$, so at least we obtain trivial algebraically quasi-solvable potentials. We will illustrate this in section 2.2 and 2.3.

- Using integrable parameterized differential equations which can be transformed into Schrödinger equations. In this case the parameter of the differential equation should coincide with the eigenvalues of H. With this method we know previously the algebraic spectrum Λ, thus, we can obtain algebraically solvable or algebraically quasi-solvable potentials depending on Λ. We will illustrate this in section 2.3.

We are interested in the spectrum (analytic spectrum) of the algebraically solvable and quasi-solvable potentials, that is, $\mathrm{spec}(H)\cap\Lambda\neq\emptyset$. For example, the potential $V(x)=|x|$ has point spectrum (see [6]) although $V(x)$ is algebraically non-solvable. Thus, when $\mathrm{spec}(H)\cap\Lambda$ is an infinite set, in the usual physical terminology these potentials are called *solvable (or exactly solvable) potentials*, see Natanzon [69]. In analogous way, when $\mathrm{spec}(H)\cap\Lambda$ is a finite set, the usual definition in physics of these potentials is *quasi-exactly solvable (or quasi-solvable) potentials* (Turbiner [102], Bender & Dunne [11], Bender & Boettcher [10], Saad et al. [87], Gibbons & Vesselov [38]).

Definition 2.1.3. Let be $\mathcal{L}$, $\widetilde{\mathcal{L}}$, pairs of linear differential equations defined over differential fields K and $\widetilde{K}$ respectively, with Picard-Vessiot extensions L and $\widetilde{L}$. Let φ be the transformation such that $\mathcal{L}\mapsto\widetilde{\mathcal{L}}$, $K\mapsto\widetilde{K}$ and $L\mapsto\widetilde{L}$, we say that:

1. φ is an *iso-Galoisian transformation* if

$$\mathrm{DGal}(L/K)=\mathrm{DGal}(\widetilde{L}/\widetilde{K}).$$

   If $\widetilde{L}=L$ and $\widetilde{K}=K$, we say that φ is a *strong iso-Galoisian transformation.*

2. φ is a *virtually iso-Galoisian transformation* if

$$(\mathrm{DGal}(L/K))^0=(\mathrm{DGal}(\widetilde{L}/\widetilde{K}))^0.$$

Remark 2.1.4. The Eigenrings of two operators $\mathfrak{L}$ and $\widetilde{\mathfrak{L}}$ are preserved under iso-Galoisian transformations.

Proposition 2.1.5. *Consider the differential equations*

$$\mathcal{L}:=\partial_x^2y+a\partial_xy+by=0,\quad \widetilde{\mathcal{L}}:=\partial_x^2\zeta=r\zeta,\quad a,\,b,\,r\,\in K.$$

Let $\kappa\in\mathbb{Q}$, $f\in K$, $a=2\kappa\partial_x(\ln f)$ and φ be the transformation such that $\mathcal{L}\mapsto\widetilde{\mathcal{L}}$. The following statements holds:

1. *φ is a strong isogaloisian transformation for $\kappa\in\mathbb{Z}$.*
2. *φ is a virtually strong isogaloisian transformation for $\kappa\in\mathbb{Q}\setminus\mathbb{Z}$.*

Proof. Assume that $\mathcal{B} = \{y_1, y_2\}$ is a basis of solutions and L is the Picard-Vessiot extension of $\mathcal{L}$, $\mathcal{B}' = \{\zeta_1, \zeta_2\}$ is a basis of solutions and $\widetilde{L}$ is the Picard-Vessiot extension of $\widetilde{\mathcal{L}}$. With the change of dependent variable $y = \zeta e^{-\frac{1}{2}\int a}$ we obtain $r = a^2/4 + \partial_x a/2 - b$ and for instance $K = \widetilde{K}$. Thus, the relationship between L and $\widetilde{L}$ depends on a:

1. If $\kappa = n \in \mathbb{Z}$, then $\mathcal{B}' = \{f^n y_1, f^n y_2\}$ which means that $L = \widetilde{L}$ and φ is strong isogaloisian.
2. If $\kappa = \frac{n}{m}$, with $\gcd(n, m) = 1$, $\frac{n}{m} \notin \mathbb{Z}$, then $\mathcal{B}' = \{f^{\frac{n}{m}} y_1, f^{\frac{n}{m}} y_2\}$ which means that $\widetilde{L}$ is either an algebraic extension of degree at most m of L, and φ is virtually strong isogaloisian, or $L = \widetilde{L}$ when $f^{\frac{n}{m}} \in K$ which means that φ is strong isogaloisian.

□

Remark 2.1.6. The transformation φ in proposition 2.1.5 is not injective, there are a lot of differential equations $\mathcal{L}$ that are transformed in the same differential equation $\widetilde{\mathcal{L}}$.

As immediate consequence of the previous proposition we have the following corollary.

Corollary 2.1.7 (Sturm-Liouville)**.** *Let $\mathcal{L}$ be the differential equation*

$$\partial_x (a \partial_x y) = (\lambda b - \mu) y, \quad a, b \in K, \quad \lambda, \mu \in \mathbb{C}$$

in where L, $\widetilde{L}$, $\widetilde{\mathcal{L}}$ and φ are given as in proposition 2.1.5. Then either $\widetilde{L}$ is a quadratic extension of L which means that φ is virtually strong isogaloisian or $\widetilde{L} = L$ when $a^{\frac{1}{2}} \in K$ which means that φ is strong isogaloisian.

2.2 Supersymmetric Quantum Mechanics with Rational Potentials

Along this section we consider as differential field $K = \mathbb{C}(x)$.

2.2.1 Polynomial Potentials

We start considering the Schrödinger equation (2.1) with polynomial potentials, i.e., $V \in \mathbb{C}[x]$, see [18, 80]. For simplicity and without lost of generality, we consider monic polynomials due to the reduced second order linear differential equation with polynomial coefficient $c_n x^n + \ldots + c_1 x + c_0$ can be transformed into the reduced second order linear differential equation with polynomial coefficient $x^n + \ldots + q_1 x + q_0$ through the change of variable $x \mapsto \sqrt[n+2]{c_n} x$.

When V is a polynomial of odd degree, is well known that the differential Galois group of the Schrödinger equation (2.1) is $\mathrm{SL}(2,\mathbb{C})$, see [56].

We present here the complete result for the Schrödinger equation (2.1) with non-constant polynomial potential (Theorem 2.2.2), see also [5, §2]. The following lemma is useful for our purposes.

Lemma 2.2.1 (Completing Squares, [5])**.** *Every even degree monic polynomial of can be written in one only way completing squares, that is,*

$$Q_{2n}(x) = x^{2n} + \sum_{k=0}^{2n-1} q_k x^k = \left(x^n + \sum_{k=0}^{n-1} a_k x^k\right)^2 + \sum_{k=0}^{n-1} b_k x^k, \tag{2.2}$$

where

$$a_{n-1} = \frac{q_{2n-1}}{2}, \quad a_{n-2} = \frac{q_{2n-2} - a_{n-1}^2}{2}, \quad a_{n-3} = \frac{q_{2n-3} - 2a_{n-1}a_{n-2}}{2}, \cdots,$$

$$a_0 = \frac{q_n - 2a_1a_{n-1} - 2a_2a_{n-2} - \cdots}{2}, \quad b_0 = q_0 - a_0^2, \quad b_1 = q_1 - 2a_0a_1, \quad \cdots,$$

$$b_{n-1} = q_{n-1} - 2a_0a_{n-1} - 2a_1a_{n-2} - \cdots.$$

Proof. See lemma 2.4 in [5, p. 275].

□

We remark that $V(x)$ as in equation (2.2) can be written in terms of the superpotential $W(x)$, i.e., $V(x) = W^2(x) - \partial_x W(x)$, when

$$nx^{n-1} + \sum_{k=1}^{n-1} ka_k x^{k-1} = -\sum_{k=0}^{n-1} b_k x^k$$

and $W(x)$ is given by

$$x^n + \sum_{k=0}^{n-1} a_k x^k.$$

The following theorem also can be found in [5, §2], see also [4]. Here we present a quantum mechanics adapted version.

Theorem 2.2.2 (Polynomial potentials and Galois groups, [5])**.** *Let us consider the Schrödinger equation* (2.1)*, with* $V(x) \in \mathbb{C}[x]$ *a polynomial of degree* $k > 0$*. Then, its differential Galois group* $\mathrm{DGal}(L_\lambda/K)$ *falls in one of the following cases:*

1. $\mathrm{DGal}(L_\lambda/K) = \mathrm{SL}(2,\mathbb{C})$,
2. $\mathrm{DGal}(L_\lambda/K) = \mathbb{B}$,

and the Eigenring of $H-\lambda$ *is trivial, i.e.,* $\mathcal{E}(H-\lambda) = \mathrm{Vect}(1)$. *Furthermore,* $\mathrm{DGal}(L_\lambda/K) = \mathbb{B}$ *if and only if the following conditions hold:*

1. $V(x)-\lambda$ *is a polynomial of degree* $k = 2n$ *writing in the form of equation (2.2).*
2. $b_{n-1}-n$ *or* $-b_{n-1}-n$ *is a positive even number* $2m$, $m\in\mathbb{Z}_+$.
3. *There exist a monic polynomial* P_m *of degree* m, *satisfying*

$$\partial_x^2 P_m + 2\left(x^n + \sum_{k=0}^{n-1} a_k x^k\right)\partial_x P_m + \left(nx^{n-1} + \sum_{k=0}^{n-2}(k+1)a_{k+1}x^k - \sum_{k=0}^{n-1} b_k x^k\right)P_m = 0,$$

or

$$\partial_x^2 P_m - 2\left(x^n + \sum_{k=0}^{n-1} a_k x^k\right)\partial_x P_m - \left(nx^{n-1} + \sum_{k=0}^{n-2}(k+1)a_{k+1}x^k + \sum_{k=0}^{n-1} b_k x^k\right)P_m = 0.$$

In such cases, the only possibilities for eigenfunctions with rational superpotentials are given by

$$\Psi_\lambda = P_m e^{f(x)}, \quad \text{or} \quad \Psi_\lambda = P_m e^{-f(x)}, \quad \text{where } f(x) = \frac{x^{n+1}}{n+1} + \sum_{k=0}^{n-1}\frac{a_k x^{k+1}}{k+1}.$$

Proof. See theorem 2.5 in [5, p. 276]. □

An easy consequence of the above theorem is the following.

Corollary 2.2.3. *Assume that* $V(x)$ *is an algebraically solvable polynomial potential. Then* $V(x)$ *is of degree* 2.

Proof. Writing $V(x)-\lambda$ in the form of equation (2.2) we see that $b_{n-1}-n = 2m$ or $-b_{n-1}-n = 2m$, where $m\in\mathbb{Z}_+$. Thus, the integrability of the Schrödinger equation with $\mathrm{Card}(\Lambda) > 1$ is obtained when b_{n-1} is constant, so $n = 1$. □

Remark 2.2.4. Given a polynomial potential $V(x)$ such that $\mathrm{spec}_p(H)\cap\Lambda\neq\emptyset$, we can obtain bound states and normalized wave functions if and only if the potential $V(x)$ is a polynomial of degree $4n+2$. Furthermore, one integrability condition of $H\Psi=\lambda\Psi$ for $\lambda\in\Lambda$ is that b_{2n} must be an odd integer. In particular, if the potential

$$V(x)=x^{4n+2n}+\mu x^{2n},\quad n>0$$

is a quasi-exactly solvable, then μ is an odd integer. For this kind of potentials, we obtain bound states only when μ is a negative odd integer.

On another hand, the non-constant polynomial potentials $V(x)$ of degree $4n$ are associated to non-hermitian Hamiltonians and $\mathcal{PT}$ invariance which are not considered here, see [10]. Furthermore, one integrability condition of $H\Psi=\lambda\Psi$ for $\lambda\in\Lambda$ is that b_{2n-1} must be an even integer. In particular, if the Schrödinger equation

$$H\Psi=\lambda\Psi,\quad V(x)=x^{4n}+\mu x^{2n-1},\quad \lambda\in\Lambda$$

is integrable, then μ is an even integer.

We present the following examples to illustrate the previous theorem and remark.

Weber's Equation and Harmonic Oscillator. The Schrödinger equation with potential $V(x)=x^2+q_1x+q_0$ corresponds to the Rehm's form of the Weber's equation (1.19), which has been studied in section 1.1.4. By lemma 2.2.1 we have

$$V(x)-\lambda=(x+a_0)^2+b_0,\quad a_0=q_1/2,\quad b_0=q_0-q_1^2/4-\lambda.$$

So that we obtain $\pm b_0-1=2m$, where $m\in\mathbb{Z}_+$. If b_0 is an odd integer, then

$$\mathrm{DGal}(L_\lambda/K)=\mathbb{B},\ \mathcal{E}(H-\lambda)=\mathrm{Vect}(1),\ \lambda\in\Lambda=\{\pm(2m+1)+q_0-q_1^2/4: m\in\mathbb{Z}_+\}$$

and the set of eigenfunctions is either

$$\Psi_\lambda=P_me^{\frac{1}{2}(x^2+q_1x)},\ \textit{or},\ \Psi_\lambda=P_me^{-\frac{1}{2}(x^2+q_1x)}.$$

In the second case we have bound state wave function and $\mathrm{spec}_p(H)\cap\Lambda=\mathrm{spec}_p(H)=\{E_m=2m+1+q_0-q_1^2/4: m\in\mathbb{Z}_+\}$, which is infinite. The polynomials P_m are related with the Hermite polynomials H_m, [23, 51, 70].

In particular we have the harmonic oscillator potential, which is given in the list (1.40) and where $H\Psi=E\Psi$. Through the change of independent variable $x\mapsto\sqrt{\frac{2}{\omega}}x$ we obtain $V(x)=x^2-1$ and $\lambda=\frac{2}{\omega}E$, that is, $q_1=0$ and $q_0=-1$. In this way $\Lambda=\{\pm(2m+1)-1: m\in\mathbb{Z}_+\}$ and the set of eigenfunctions is either

$$\Psi_\lambda=P_me^{\frac{1}{2}x^2},\ \textit{or},\ \Psi_\lambda=P_me^{-\frac{1}{2}x^2},$$

where as below, $\mathrm{DGal}(L_\lambda/K)=\mathbb{B}$ and $\mathcal{E}(H-\lambda)=\{1\}$ for all $\lambda\in\Lambda$. In the second case we have bound state wave function, $\mathrm{spec}_p(H)\cap\Lambda=\mathrm{spec}_p(H)=\Lambda_+=\{2m: m\in\mathbb{Z}_+\}$ and $P_m=H_m$. The wave functions of $H\Psi=E\Psi$ for the harmonic oscillator potential are given by

$$\Psi_m=H_m\left(\sqrt{\frac{2}{\omega}}x\right)\Psi_0,\quad \Psi_0=e^{-\frac{\omega}{4}x^2},\quad E=E_m=m\omega.$$

Quartic and Sextic Anharmonic Oscillator. The Schrödinger equation with potential $V(x)=x^4+q_3x^3+q_2x^2+q_1x+q_0$ can be obtained through transformations of confluent Heun's equation, which is not considered here. By lemma 2.2.1 we have

$$V(x)-\lambda=(x^2+a_1x+a_0)^2+b_1x+b_0,$$

where $a_1=q_3/2$, $a_0=q_2/2-a_1^2/2$, $b_1=q_1-2a_0a_1$ and $b_0=q_0-a_0^2-\lambda$. So that we obtain $\pm b_1-2=2m$, where $m\in\mathbb{Z}_+$. If $\Lambda\neq\emptyset$, then b_1 is an even integer, P_m satisfy the relation (1.6) and $\mathrm{DGal}(L_\lambda/K)=\mathbb{B}$ for all $\lambda\in\Lambda$. The set of eigenfunctions is either

$$\Psi_\lambda=P_me^{\frac{x^3}{3}+\frac{a_1x^2}{2}+a_0x},\ \textit{or},\ \Psi_\lambda=P_me^{-\left(\frac{x^3}{3}+\frac{a_1x^2}{2}+a_0x\right)},$$

where λ and m are related, which means that Λ is finite, i.e., the potential is algebraically quasi-solvable. In particular for $q_3=2\imath l$, $q_2=l^2-2k$, $q_1=2\imath(lk-J)$ and $q_0=0$, we have the quartic anharmonic oscillator potential, which can be found in [10].

Now, considering the potentials $V(x,\mu)=x^4+4x^3+2x^2-\mu x$, again by lemma 2.2.1 we have that

$$V(x,\mu)-\lambda=(x^2+2x-1)^2+(4-\mu)x-1-\lambda,$$

so that $\pm(4-\mu)-2=2n$, where $n\in\mathbb{Z}_+$ and in consequence $\mu\in2\mathbb{Z}$. Such μ can be either $\mu=2-2n$ or $\mu=2n+6$, where $n\in\mathbb{Z}_+$. By theorem 2.2.2, there exists a monic polynomial P_n satisfying respectively

$$\partial_x^2P_n+(2x^2+4x-2)\partial_xP_n+((\mu-2)x+3+\lambda)P_n=0,\quad \mu=2-2n,\quad \text{or}$$

$$\partial_x^2P_n-(2x^2+4x-2)\partial_xP_n+((\mu-6)x-1+\lambda)P_n=0,\quad \mu=2n+6$$

for $\Lambda\neq\emptyset$. This algebraic relation between the coefficients of P_n, μ and λ give us the set Λ in the following way:

1. Write $P_n=x^n+c_{n-1}x^{n-1}+\ldots+c_0$, where c_i are unknown.
2. Pick μ and replace P_n in the algebraic relation (1.6) to obtain a polynomial of degree n with $n+1$ undetermined coefficients involving $c_0,\ldots c_{n-1}$ and λ. Each of such coefficients must be zero.

3. The term $n+1$ is linear in λ and c_{n-1}, thus we write c_{n-1} in terms of λ. After of the elimination of the term $n+1$, we replace c_{n-1} in the term n to obtain a quadratic polynomial in λ and so on until arrive to the constant term which is a polynomial of degree $n+1$ in λ ($Q_{n+1}(\lambda)$). In this way, $\Lambda = \{\lambda : Q_{n+1}(\lambda) = 0\}$ and $c_0, \dots, c_{n-1}$ are determined for each value of λ.

   For $\mu = 2n+6$, we have:

$$\begin{array}{llll} n=0, & V(x,6), & P_0 = 1, & \Lambda = \{1\} \\ n=1, & V(x,8), & P_1 = x+1 \mp \sqrt{2}, & \Lambda = \{3 \pm 2\sqrt{2}\} \\ \vdots & & & \end{array}$$

and the set of eigenfunctions is

$$\Psi_{\lambda,\mu} = P_n e^{-\frac{1}{3}x^3 - x^2 + x}.$$

In the same way, we can obtain Λ, P_n and $\Psi_{\lambda,\mu}$ for $\mu = 2-2n$. However, we have not bound states, $\mathrm{spec}_p(H) \cap \Lambda = \emptyset$, $\mathrm{DGal}(L_\lambda/K) = \mathbb{B}$ and $\mathcal{E}(H-\lambda) = \mathrm{Vect}(1)$ for all $\lambda \in \Lambda$.

The well known *sextic anharmonic oscillator* $x^6 + q_5 x^5 + \cdots + q_1 x + q_0$ can be treated in a similar way, obtaining bound states wave functions and the Bender-Dunne orthogonal polynomials, which corresponds to $Q_{n+1}(\lambda)$, i.e., we can have the same results of [11, 38, 87]. The Schrödinger equation with this potential, under suitable transformations, also falls in a confluent Heun's equation.

2.2.2 Rational Potentials and Kovacic's Algorithm

In this subsection we apply Kovacic's algorithm to solve the Schrödinger equation with rational potentials listed in the previous chapter (equation (1.40)).

Three dimensional harmonic oscillator potential:

$$V(r) = \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+1)}{r^2} - \left(\ell + \frac{3}{2}\right)\omega, \quad \ell \in \mathbb{Z},$$

we can see that Schrödinger equation (equation (1.22)) for this case can be written as

$$\partial_r^2 \Psi = \left(\left(\frac{1}{2}\omega r\right)^2 + \frac{\ell(\ell+1)}{r^2} - \left(\ell + \frac{3}{2}\right)\omega - E\right)\Psi.$$

By the change $r \mapsto \left(\sqrt{\frac{2}{\omega}}\right) r$ we obtain the Schrödinger equation

$$\partial_r^2 \Psi = \left(r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell+3) - \lambda\right)\Psi, \quad \lambda = \frac{2}{\omega}E$$

and in order to apply Kovacic's algorithm, we denote:

$$R = r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell+3) - \lambda.$$

We can see that this equation could fall in case 1, in case 2 or in case 4 (of Kovacic's algorithm). We start discarding the case 2 because by step 1 (of Kovacic's algorithm) we should have conditions c_2 and ∞_3, in this way we should have $E_c = \{2, 4+4\ell, -4\ell\}$ and $E_\infty = \{-2\}$, and by step 2, we should have that $n = -4 \notin \mathbb{Z}_+$, so that $D = \emptyset$, that is, this Schrödinger equation never falls in case 2. Now, we only work with case 1; by step 1, conditions c_2 and ∞_3 are satisfied, so that

$$\left[\sqrt{R}\right]_c = 0, \quad \alpha_c^\pm = \frac{1 \pm (2\ell+1)}{2}, \quad \left[\sqrt{R}\right]_\infty = r, \quad \alpha_\infty^\pm = \frac{\mp(\lambda+2\ell+3)-1}{2}.$$

By step 2 we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\begin{array}{lll}
\Lambda_{++}) & n = \alpha_\infty^+ - \alpha_0^+ = -\frac{1}{2}\left(4\ell+6+\lambda\right), & \lambda = -2n-4\ell-6, \\[2ex]
\Lambda_{+-}) & n = \alpha_\infty^+ - \alpha_0^- = -\frac{1}{2}\left(4+\lambda\right), & \lambda = -2n-4, \\[2ex]
\Lambda_{-+}) & n = \alpha_\infty^- - \alpha_0^+ = \frac{\lambda}{2}, & \lambda = 2n, \\[2ex]
\Lambda_{--}) & n = \alpha_\infty^- - \alpha_0^- = \frac{1}{2}\left(4\ell+2+\lambda\right), & \lambda = 2n-4\ell-2,
\end{array}$$

where $\Lambda_{++} \cup \Lambda_{+-} \cup \Lambda_{-+} \cup \Lambda_{--} = \Lambda$, which means that $\lambda = 2m$, $m \in \mathbb{Z}$. Now, for $\lambda \in \Lambda$, the rational function ω in Kovacic's algorithm is given by:

$$\begin{array}{lll}
\Lambda_{++}) & \omega = r + \frac{\ell+1}{r}, & R_n = r^2 + \frac{\ell(\ell+1)}{r^2} + (2\ell+3) + 2n, \\[2ex]
\Lambda_{+-}) & \omega = r - \frac{\ell}{r}, & R_n = r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell-1) + 2n, \\[2ex]
\Lambda_{-+}) & \omega = -r + \frac{\ell+1}{r}, & R_n = r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell+3) - 2n, \\[2ex]
\Lambda_{--}) & \omega = -r - \frac{\ell}{r}, & R_n = r^2 + \frac{\ell(\ell+1)}{r^2} + (2\ell-1) - 2n,
\end{array}$$

where R_n is the coefficient of the differential equation $\widetilde{\mathcal{L}}_n := \partial_r^2\Psi = R_n\Psi$, which is integrable for every n and for every $\lambda \in \Lambda$ we can see that $\mathrm{DGal}(\widetilde{L}_n/K) = \mathrm{DGal}(L_\lambda/K)$, where $\mathcal{L}_\lambda := H\Psi = \lambda\Psi$ and $\lambda \in \Lambda$.

By step 3, there exists a polynomial of degree n satisfying the relation (1.6):

$$\begin{array}{llcll}
\Lambda_{++}) & \partial_r^2 P_n + 2\left(r + \frac{\ell+1}{r}\right)\partial_r P_n - 2nP_n & = & 0, & \lambda \in \Lambda_-, \\[2ex]
\Lambda_{+-}) & \partial_r^2 P_n + 2\left(r - \frac{\ell}{r}\right)\partial_r P_n - 2nP_n & = & 0, & \lambda \in \Lambda_-, \\[2ex]
\Lambda_{-+}) & \partial_r^2 P_n + 2\left(-r + \frac{\ell+1}{r}\right)\partial_r P_n + 2nP_n & = & 0, & \lambda \in \Lambda_+, \\[2ex]
\Lambda_{--}) & \partial_r^2 P_n + 2\left(-r - \frac{\ell}{r}\right)\partial_r P_n + 2nP_n & = & 0, & \lambda \in \Lambda.
\end{array}$$

These polynomials exists for all $\lambda \in \Lambda$ when their degrees are $n \in 2\mathbb{Z}$, while for $n \in 2\mathbb{Z}+1$, they exists only for the cases Λ_{-+}) and Λ_{--}) with special conditions. In this way, we have obtained the algebraic spectrum $\Lambda = 2\mathbb{Z}$, where $\Lambda_{++} = 4\mathbb{Z}_{-}$, $\Lambda_{+-} = 2\mathbb{Z}_{-}$, $\Lambda_{-+} = 4\mathbb{Z}_{+}$, $\Lambda_{--} = 2\mathbb{Z}$.

The possibilities for eigenfunctions, considering only $\lambda \in 4\mathbb{Z}$, are given by

$$\begin{array}{lrcll} \Lambda_{++}) & \Psi_n(r) & = & r^{\ell+1}P_{2n}(r)e^{\frac{r^2}{2}}, & \lambda \in \Lambda_{-}, \\ \Lambda_{+-}) & \Psi_n(r) & = & r^{-\ell}P_{2n}(r)e^{\frac{r^2}{2}}, & \lambda \in \Lambda_{-}, \\ \Lambda_{-+}) & \Psi_n(r) & = & r^{\ell+1}P_{2n}(r)e^{\frac{-r^2}{2}}, & \lambda \in \Lambda_{+}, \\ \Lambda_{--}) & \Psi_n(r) & = & r^{-\ell}P_{2n}(r)e^{\frac{-r^2}{2}}, & \lambda \in \Lambda. \end{array}$$

To obtain the point spectrum, we look Ψ_n satisfying the bound state conditions (equation (1.23)) which is in only true for $\lambda \in \Lambda_{-+}$. With the change $r \mapsto \sqrt{\frac{\omega}{2}}r$, the point spectrum and ground state of the Schrödinger equation with the 3D-harmonic oscillator potential are respectively $\mathrm{spec}_p(H) = \{E_n : n \in \mathbb{Z}_+\}$, where $E_n = 2n\omega$, being ω the angular velocity, and

$$\Psi_0 = \left(\sqrt{\frac{\omega}{2}}r\right)^{\ell+1} e^{-\frac{\omega}{4}r^2}.$$

The bound state wave functions are obtained as $\Psi_n = P_{2n}\Psi_0$. Now, we can see that $\mathrm{DGal}(L_0/K) = \mathbb{B}$ and $\mathcal{E}(H) = \mathrm{Vect}(1)$. Since $\Psi_n = P_{2n}\Psi_0$, for all $\lambda \in \Lambda$ we have that $\mathrm{DGal}(L_\lambda/K) = \mathbb{B}$ and $\mathcal{E}(H-\lambda) = \mathrm{Vect}(1)$. In particular, $\mathrm{DGal}(L_\lambda/K) = \mathbb{B}$ and $\mathcal{E}(H-\lambda) = \mathrm{Vect}(1)$ for all $\lambda \in \mathrm{spec}_p(H)$, where $\lambda = \frac{2}{\omega}E$.

We remark that the Schrödinger equation with the 3D-harmonic oscillator potential, through the changes $r \mapsto \frac{1}{2}\omega r^2$ and $\Psi \mapsto \sqrt{r}\Psi$, fall in a Whittaker differential equation (equation (1.16)) in where the parameters are given by

$$\kappa = \frac{(2\ell+3)\omega + 2E}{4\omega}, \quad \mu = \frac{1}{2}\ell + \frac{1}{4}.$$

Applying theorem 1.1.31, we can see that for integrability, $\pm\kappa \pm \mu$ must be a half integer. These conditions coincides with our four sets $\Lambda_{\pm\pm}$.

Coulomb potential:

$$V(r) = -\frac{\mathrm{e}^2}{r} + \frac{\ell(\ell+1)}{r^2} + \frac{\mathrm{e}^4}{4(\ell+1)^2}, \quad \ell \in \mathbb{Z}.$$

we can see that the Schrödinger equation (equation (1.22)) for this case can be written as

$$\partial_r^2\Psi = \left(\frac{\ell(\ell+1)}{r^2} - \frac{\mathrm{e}^2}{r} + \frac{\mathrm{e}^4}{4(\ell+1)^2} - E\right)\Psi.$$

By the change $r \mapsto \frac{2(\ell+1)}{\mathrm{e}^2} r$ we obtain the Schrödinger equation

$$\partial_r^2 \Psi = \left(\frac{\ell(\ell+1)}{r^2} - \frac{2(\ell+1)}{r} + 1 - \lambda\right)\Psi, \quad \lambda = \frac{4(\ell+1)^2}{\mathrm{e}^4}E$$

and in order to apply Kovacic algorithm, we denote

$$R = \frac{\ell(\ell+1)}{r^2} - \frac{2(\ell+1)}{r} + 1 - \lambda.$$

Firstly we analyze the case for $\lambda = 1$: we can see that this equation only could fall in case 2 or in case 4 of Kovacic's algorithm. We start discarding the case 2 because by step 1 we should have conditions c_2 and ∞_3. In this way we should have $E_c = \{2, 4+4\ell, -4\ell\}$ and $E_\infty = \{1\}$ and by step 2, we should have that $n \notin \mathbb{Z}$. Thus, $n \notin \mathbb{Z}_+$ and $D = \emptyset$, that is, the differential Galois group of this Schrödinger equation for $\lambda = 1$ is $\mathrm{SL}(2, \mathbb{C})$.

Now, we analyze the case for $\lambda \neq 1$: we can see that this equation could fall in case 1, in case 2 or in case 4. We start discarding the case 2 because by step 1 we should have conditions c_2 and ∞_3, so that we should have $E_c = \{2, 4+4\ell, -4\ell\}$ and $E_\infty = \{0\}$. By step 2, we should have that $n = 2\ell \in \mathbb{Z}_+$, so that $D = \{2\ell\}$ and the rational function θ is $\theta = \frac{-2\ell}{x}$, but we discard this case because only could exists one polynomial of degree 2ℓ for a fixed ℓ, and for instance, only could exist one eigenstate and one eigenfunction for the Schrödinger equation.

Now, we only work with case 1, by step 1, conditions c_2 and ∞_3 are satisfied. Thus,

$$\left[\sqrt{R}\right]_c = 0, \quad \alpha_c^{\pm} = \frac{1 \pm (2\ell+1)}{2}, \quad \left[\sqrt{R}\right]_\infty = \sqrt{1-\lambda}, \quad \alpha_\infty^{\pm} = \mp\frac{\ell+1}{\sqrt{1-\lambda}}.$$

By step 2 we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$\Lambda_{++})$ $\quad n = \alpha_\infty^+ - \alpha_0^+ = -(\ell+1)\left(1 + \frac{1}{\sqrt{1-\lambda}}\right), \quad \lambda = 1 - \left(\frac{\ell+1}{\ell+1+n}\right)^2,$

$\Lambda_{+-})$ $\quad n = \alpha_\infty^+ - \alpha_0^- = -\frac{\ell+1}{\sqrt{1-\lambda}} + \ell, \quad \lambda = 1 - \left(\frac{\ell+1}{\ell-n}\right)^2,$

$\Lambda_{-+})$ $\quad n = \alpha_\infty^- - \alpha_0^+ = (\ell+1)\left(\frac{1}{\sqrt{1-\lambda}} - 1\right), \quad \lambda = 1 - \left(\frac{\ell+1}{\ell+1+n}\right)^2,$

$\Lambda_{--})$ $\quad n = \alpha_\infty^- - \alpha_0^- = \frac{\ell+1}{\sqrt{1-\lambda}} + \ell, \quad \lambda = 1 - \left(\frac{\ell+1}{\ell-n}\right)^2.$

We can see that $\lambda \in \Lambda_-$ when $\lambda \leq 0$, while $\lambda \in \Lambda_+$ when $0 \leq \lambda < 1$. Furthermore:

$$\Lambda_{++}) \qquad \ell \leq -1, \qquad \lambda \in \begin{cases} \Lambda_-, & \ell \leq \frac{-n-2}{2} \\ \Lambda_+, & \frac{-n-2}{2} \leq \ell \leq -1 \end{cases}$$

$$\Lambda_{+-}) \qquad \ell > 0, \qquad \lambda \in \begin{cases} \Lambda_-, & \ell \geq \frac{n-1}{2} \\ \Lambda_+, & 0 \leq \ell \leq \frac{n-1}{2} \end{cases}$$

$$\Lambda_{-+}) \qquad \ell \in \mathbb{Z}, \qquad \lambda \in \begin{cases} \Lambda_-, & \ell \leq \frac{-n-2}{2} \\ \Lambda_+, & \ell \geq -1 \end{cases}$$

$$\Lambda_{--}) \qquad \ell > 0, \qquad \lambda \in \begin{cases} \Lambda_-, & \ell \geq \frac{n-1}{2} \\ \Lambda_+, & 0 \leq \ell \leq \frac{n-1}{2} \end{cases}$$

In this way, the possible algebraic spectrum can be $\Lambda = \Lambda_{++} \cup \Lambda_{+-} \cup \Lambda_{-+} \cup \Lambda_{--}$, that is

$$\Lambda = \left\{ 1 - \left(\frac{\ell+1}{\ell+1+n} \right)^2 : n \in \mathbb{Z}_+ \right\} \cup \left\{ 1 - \left(\frac{\ell+1}{\ell-n} \right)^2 : n \in \mathbb{Z}_+ \right\}, \tag{2.3}$$

Now, for $\lambda \in \Lambda$, the rational function ω is given by:

$$\Lambda_{++}) \qquad \omega = \tfrac{\ell+1}{\ell+1+n} + \tfrac{\ell+1}{r}, \qquad \lambda \in \Lambda_{++}, \quad R_n = \tfrac{\ell(\ell+1)}{r^2} - \tfrac{2(\ell+1)}{r} + \left(\tfrac{\ell+1}{\ell+1+n}\right)^2,$$

$$\Lambda_{+-}) \qquad \omega = \tfrac{\ell+1}{\ell-n} - \tfrac{\ell}{r}, \qquad \lambda \in \Lambda_{+-}, \quad R_n = \tfrac{\ell(\ell+1)}{r^2} - \tfrac{2(\ell+1)}{r} + \left(\tfrac{\ell+1}{\ell-n}\right)^2,$$

$$\Lambda_{-+}) \qquad \omega = -\tfrac{\ell+1}{\ell+1+n} + \tfrac{\ell+1}{r}, \qquad \lambda \in \Lambda_{-+}, \quad R_n = \tfrac{\ell(\ell+1)}{r^2} - \tfrac{2(\ell+1)}{r} + \left(\tfrac{\ell+1}{\ell+1+n}\right)^2,$$

$$\Lambda_{--}) \qquad \omega = -\tfrac{\ell+1}{\ell-n} - \tfrac{\ell}{r}, \qquad \lambda \in \Lambda_{--}, \quad R_n = \tfrac{\ell(\ell+1)}{r^2} - \tfrac{2(\ell+1)}{r} + \left(\tfrac{\ell+1}{\ell-n}\right)^2,$$

where R_n is the coefficient of the differential equation $\partial_r^2 \Psi = R_n \Psi$, which is integrable for every n.

By step 3, there exists a polynomial of degree n satisfying the relation (1.6),

$$\begin{array}{lll}
\Lambda_{++}) & \partial_r^2 P_n + 2\left(\frac{\ell+1}{\ell+1+n} + \frac{\ell+1}{r}\right)\partial_r P_n + \frac{2(\ell+1)}{r}\left(1 + \frac{\ell+1}{\ell+1+n}\right)P_n & = 0, \\
\Lambda_{+-}) & \partial_r^2 P_n + 2\left(\frac{\ell+1}{\ell-n} - \frac{\ell}{r}\right)\partial_r P_n + \frac{2(\ell+1)}{r}\left(1 - \frac{\ell+1}{\ell-n}\right)P_n & = 0, \\
\Lambda_{-+}) & \partial_r^2 P_n + 2\left(-\frac{\ell+1}{\ell+1+n} + \frac{\ell+1}{r}\right)\partial_r P_n + \frac{2(\ell+1)}{r}\left(1 - \frac{\ell+1}{\ell+1+n}\right)P_n & = 0, \\
\Lambda_{--}) & \partial_r^2 P_n + 2\left(-\frac{\ell+1}{\ell-n} - \frac{\ell}{r}\right)\partial_r P_n + \frac{2(\ell+1)}{r}\left(1 + \frac{\ell+1}{\ell-n}\right)P_n & = 0.
\end{array}$$

These polynomials exists for every $\lambda \in \Lambda$ when $n \in \mathbb{Z}$, but $P_0 = 1$ is satisfied only for $\lambda \in \Lambda_{-+}$. In this way, we have confirmed that the algebraic spectrum Λ is given by equation (2.3).

The possibilities for eigenfunctions are given by

$$\begin{array}{lllll}
\Lambda_{++}) & \Psi_n(r) = r^{\ell+1}P_n(r)f_n(r)e^{r}, & f_n(r) = e^{\frac{-nr}{\ell+1+n}}, & \lambda \in \begin{cases} \Lambda_-, & \ell \leq \frac{-n-2}{2} \\ \Lambda_+, & \frac{-n-2}{2} \leq \ell \leq -1 \end{cases} \\
\Lambda_{+-}) & \Psi_n(r) = r^{-\ell}P_n(r)f_n(r)e^{r}, & f_n(r) = e^{\frac{n+1}{\ell-n}r}, & \lambda \in \begin{cases} \Lambda_-, & \ell \geq \frac{n-1}{2} \\ \Lambda_+, & 0 \leq \ell \leq \frac{n-1}{2} \end{cases} \\
\Lambda_{-+}) & \Psi_n(r) = r^{\ell+1}P_n(r)f_n(r)e^{-r}, & f_n(r) = e^{\frac{nr}{\ell+1+n}}, & \lambda \in \begin{cases} \Lambda_-, & \ell \leq \frac{-n-2}{2} \\ \Lambda_+, & \ell \geq -1 \end{cases} \\
\Lambda_{--}) & \Psi_n(r) = r^{-\ell}P_n(r)f_n(r)e^{-r}, & f_n(r) = e^{\frac{n+1}{n-\ell}r}, & \lambda \in \begin{cases} \Lambda_-, & \ell \geq \frac{n-1}{2} \\ \Lambda_+, & 0 \leq \ell \leq \frac{n-1}{2} \end{cases}
\end{array}$$

but Ψ_n should satisfy the bound state conditions (equation (1.23)) which is only true for $\lambda \in \Lambda_{-+} \cap \Lambda_+$, so that we choose $\Lambda_{-+} \cap \Lambda_+ = \mathrm{spec}_p(H)$, that is

$$\mathrm{spec}_p(H) = \left\{1 - \left(\frac{\ell+1}{\ell+n+1}\right)^2 : \quad n \in \mathbb{Z}_+, \quad \ell \geq -1\right\}.$$

By the change $r \mapsto \frac{\mathrm{e}^2}{2(\ell+1)}r$, the point spectrum and ground state of the Schrödinger equation with Coulomb potential are respectively

$$\mathrm{spec}_p(H) = \{E_n : n \in \mathbb{Z}_+\}, \quad E_n = \frac{\mathrm{e}^4}{4}\left(\frac{1}{(\ell+1)^2} - \frac{1}{(\ell+1+n)^2}\right)$$

and

$$\Psi_0 = \left(\frac{\mathrm{e}^2}{2(\ell+1)}r\right)^{\ell+1} e^{-\frac{\mathrm{e}^2}{2(\ell+1)}r}.$$

The eigenstates are given by $\Psi_n = P_n f_n \Psi_0$, where

$$f_n(r) = e^{\frac{ne^2 r}{2(\ell+1+n)(\ell+1)}}.$$

Now, we can see that $\mathrm{DGal}(L_0/K) = \mathbb{B}$ and $\mathcal{E}(H) = \mathrm{Vect}(1)$. Since $\Psi_n = P_{2n} f_n \Psi_0$, for all $\lambda \in \Lambda$ we have that $\mathrm{DGal}(L_\lambda/K) = \mathbb{B}$ and $\mathcal{E}(H-\lambda) = \mathrm{Vect}(1)$. In particular, $\mathrm{DGal}(L_\lambda/K) = \mathbb{B}$ and $\mathcal{E}(H-\lambda) = \mathrm{Vect}(1)$ for all $E \in \mathrm{spec}_p(H)$, where $E = \frac{e^4}{4(\ell+1)^2}\lambda$.

We remark that, as in the three dimensional harmonic oscillator, the Schrödinger equation with the Coulomb potential, through the change

$$r \mapsto \frac{\sqrt{-4\,(\ell+1)^2\,E + e^4}}{\ell+1} r,$$

falls in a Whittaker differential equation (equation (1.16)) in where the parameters are given by

$$\kappa = \frac{e^2\,(\ell+1)}{\sqrt{-4\,(\ell+1)^2\,E + e^4}}, \quad \mu = \ell + \frac{1}{2}.$$

Applying theorem 1.1.31, we can impose $\pm\kappa \pm \mu$ half integer, to coincides with our four sets $\Lambda_{\pm\pm}$.

Remark 2.2.5. By direct application of Kovacic's algorithm we have:

- The Schrödinger equation (2.1) with potential

  $$V(x) = ax^2 + \frac{b}{x^2}$$

  is integrable for $\lambda \in \Lambda$ when

  - $a = 0, \quad b = \mu(\mu+1), \quad \mu \in \mathbb{C}, \quad \Lambda = \mathbb{C}$,
  - $a = 1, \quad b = 0, \quad \lambda \in \Lambda = 2\mathbb{Z}+1$,
  - $a = 1, \quad b = \ell(\ell+1), \quad \ell \in \mathbb{Z}^*, \Lambda = 2\mathbb{Z}+1$.

- The only rational potentials (up to transformations) in which the elements of the algebraic spectrum are placed at the same distance, belongs to the family of potentials given by

  $$V(x) = \sum_{k=-\infty}^{2} a_k x^k, \quad a_2 \neq 0.$$

  In particular, the set Λ for the harmonic oscillator ($a = 1$, $b = 0$) and $3D$ harmonic oscillator ($a = 1$, $b = \ell(\ell+1)$) satisfies this.

Proposition 2.2.6. *Let $\mathcal{L}_\lambda$ be the Schrödinger equation* (2.1) *with $K=\mathbb{C}(x)$ and Picard-Vessiot extension L_λ. If* $\mathrm{DGal}(L_0/K)$ *is finite primitive, then* $\mathrm{DGal}(L_\lambda/K)$ *is not finite primitive for all $\lambda\in\Lambda\setminus\{0\}$.*

Proof. Pick $V\in\mathbb{C}(x)$ such that $\mathcal{L}_0$ falls in case 3 of Kovacic algorithm, then $\circ u_\infty \geq 2$. Assume $t,s\in\mathbb{C}[x]$ such that $V=\frac{s}{t}$, then $\deg(t)\geq\deg(s)+2$ and $V-\lambda=\frac{s-\lambda t}{t}$. Now, for $\lambda\neq 0$ we have that $\deg(s-\lambda t)=\deg(t)$ and therefore $\circ(V-\lambda)_\infty=0$. So that for $\lambda\neq 0$, the equation $\mathcal{L}_\lambda$ does not falls in case 3 of Kovacic algorithm and therefore $\mathrm{DGal}(L_\lambda/K)$ is not finite primitive. □

Corollary 2.2.7. *Let $\mathcal{L}_\lambda$ be the Schrödinger equation* (2.1) *with $K=\mathbb{C}(x)$ and Picard-Vessiot extension L_λ. If* $\mathrm{Card}(\Lambda)>1$, *then there is either zero or one value of λ such that* $\mathrm{DGal}(L_\lambda/K)$ *is a finite primitive group.*

Proof. Assume that $\mathrm{Card}(\Lambda)>1$. Thus, by proposition 2.2.6, the Schrödinger equation does not falls in case 3 of Kovacic's algorithm. □

It seems that the study of the differential Galois groups of the Schrödinger equation with the Coulomb potential has been analyzed by Jean-Pierre Ramis using his summability theory in the eighties of the past century, see [82].

2.2.3 Darboux Transformations

Here we present a Galoisian approach to Darboux transformation, Crum iteration and shape invariant potentials. We denote by $W(y_1,\ldots,y_n)$ the Wronskian

$$W(y_1,\ldots,y_n)=\begin{vmatrix} y_1 & \cdots & y_n\\ \vdots & & \vdots\\ \partial_x^{n-1}y_1 & \cdots & \partial_x^{n-1}y_n\end{vmatrix},$$

by DT the Darboux transformation, by DT_n the n iteration of DT and by CI_n the Crum iteration. Also we use the notation of subsection 1.2. We recall that $K=\mathbb{C}(x)$ and in case of other differential fields, as usually is considered along this thesis, we mean the smallest differential containing the coefficients of the linear differential equations.

Theorem 2.2.8 (Galoisian version of DT)**.** *Assume $H_\pm=\partial_x^2+V_\pm(x)$ and $\Lambda\neq\emptyset$. Let $\mathcal{L}_\lambda$ given by the Schrödinger equation $H_-\Psi^{(-)}=\lambda\Psi^{(-)}$ with $V_-(x)\in K$ and $\widetilde{\mathcal{L}}_\lambda$ given by the Schrödinger equation $H_+\Psi^{(+)}=\lambda\Psi^{(+)}$ with $V_+(x)\in\widetilde{K}$. Let* DT *be the transformation such that $\mathcal{L}\mapsto\widetilde{\mathcal{L}}$, $V_-\mapsto V_+$, $\Psi^{(-)}\mapsto\Psi^{(+)}$. Then the following statements holds:*

i) $\mathrm{DT}(V_-)=V_+=\Psi_{\lambda_1}^{(-)}\partial_x^2\left(\frac{1}{\Psi_{\lambda_1}^{(-)}}\right)+\lambda_1=V_--2\partial_x^2(\ln\Psi_{\lambda_1}^{(-)})$,

$\mathrm{DT}(\Psi_{\lambda_1}^{(-)})=\Psi_{\lambda_1}^{(+)}=\frac{1}{\Psi_{\lambda_1}^{(-)}}$, *where $\Psi_{\lambda_1}^{(-)}$ is a particular solution of $\mathcal{L}_{\lambda_1}$, $\lambda_1\in\Lambda$.*

ii) $\mathrm{DT}(\Psi_\lambda^{(-)}) = \Psi_\lambda^{(+)} = \partial_x\Psi_\lambda^{(-)} - \partial_x(\ln\Psi_{\lambda_1}^{(-)})\Psi_\lambda^{(-)} = \frac{W(\Psi_{\lambda_1}^{(-)},\Psi_\lambda^{(-)})}{W(\Psi_{\lambda_1}^{(-)})}$, $\lambda \neq \lambda_1$, *where* $\Psi_\lambda^{(-)}$ *is the general solution of* $\mathcal{L}_\lambda$ *for* $\lambda \in \Lambda \setminus \{\lambda_1\}$ *and* $\Psi_\lambda^{(+)}$ *is the general solution of* $\widetilde{\mathcal{L}}_\lambda$ *also for* $\lambda \in \Lambda \setminus \{\lambda_1\}$.

In agreement with the previous theorem we obtain the following results.

Proposition 2.2.9. DT *is isogaloisian and virtually strong isogaloisian. Furthermore, if* $\partial_x(\ln\Psi_{\lambda_1}^{(-)}) \in K$, *then* DT *is strong isogaloisian.*

Proof. Let K, L_λ be the differential field and the Picard-Vessiot extension of the equation $\mathcal{L}_\lambda$. Let $\widetilde{K}$, $\widetilde{L}_\lambda$ the differential field and the Picard-Vessiot extension of the equation $\widetilde{\mathcal{L}}_\lambda$. Due to $\mathrm{DT}(V_-) = V_+ = 2W^2 - V_- - 2\lambda_1$, where $W = -\partial_x(\ln\Psi_{\lambda_1}^{(-)})$, we have $\widetilde{K} = K\left\langle \partial_x(\ln\Psi_{\lambda_1}^{(-)})\right\rangle$. By theorem 1.1.10 we have that the Riccati equation $\partial_x W = V_- - W^2$ has one algebraic solution, in this case $W = -\partial_x(\ln\Psi_{\lambda_1}^{(-)})$. Let $\langle\Psi_{(1,\lambda)}^{(-)}, \Psi_{(2,\lambda)}^{(-)}\rangle$ be a basis of solutions for equation $\mathcal{L}_\lambda$ and $\langle\Psi_{(1,\lambda)}^{(+)}, \Psi_{(2,\lambda)}^{(+)}\rangle$ a basis of solutions for equation $\widetilde{\mathcal{L}}_\lambda$. Since the differential field for equation $\widetilde{\mathcal{L}}_\lambda$ is $\widetilde{K} = K\langle\partial_x(\ln\Psi_{\lambda_1}^{(-)})\rangle$, we have that $L = K\langle\Psi_{(1,\lambda)}^{(-)}, \Psi_{(2,\lambda)}^{(-)}\rangle$ and

$$\widetilde{L} = \widetilde{K}\langle\Psi_{(1,\lambda)}^{(+)}, \Psi_{(2,\lambda)}^{(+)}\rangle = K\langle\Psi_{(1,\lambda)}^{(+)}, \Psi_{(2,\lambda)}^{(+)}, \partial_x(\ln\Psi_{\lambda_1}^{(-)})\rangle$$

$$= K\langle\Psi_{(1,\lambda)}^{(-)}, \Psi_{(2,\lambda)}^{(-)}, \partial_x(\ln\Psi_{\lambda_1}^{(-)})\rangle = \widetilde{K}\langle\Psi_{(1,\lambda)}^{(-)}, \Psi_{(2,\lambda)}^{(-)}\rangle,$$

for $\lambda = \lambda_1$ and for $\lambda \neq \lambda_1$. Since $\partial_x(\ln\Psi_{\lambda_1}^{(-)})$ is algebraic over K, then

$$(\mathrm{DGal}(L_\lambda/K))^0 = (\mathrm{DGal}(\widetilde{L}_\lambda/K))^0, \quad \mathrm{DGal}(L_\lambda/K) = \mathrm{DGal}(\widetilde{L}_\lambda/\widetilde{K}),$$

which means that DT is a virtually strong and isogalosian transformation.
In the case $\partial_x(\ln\Psi_{\lambda_1}^{(-)}) \in K$, then $\widetilde{K} = K$ and $\widetilde{L} = L$, which means that DT is a strong isogalosian transformation. □

Proposition 2.2.10. *Consider* $\mathfrak{L}_\lambda := H_- - \lambda$ *and* $\widetilde{\mathfrak{L}}_\lambda := H_+ - \lambda$ *such that* $\mathrm{DT}(H_- - \lambda) = H_+ - \lambda$. *The eigenrings of* $\mathfrak{L}_\lambda$ *and* $\widetilde{\mathfrak{L}}_\lambda$ *are isomorphic.*

Proof. Assume $\mathcal{E}(\mathfrak{L}_\lambda)$ and $\mathcal{E}(\widetilde{\mathfrak{L}}_\lambda)$ the eigenrings of $\mathfrak{L}_\lambda$ and $\widetilde{\mathfrak{L}}_\lambda$ respectively. By proposition 2.2.9 the connected identity component of the Galois group is preserved by Darboux transformation and for instance the eigenrings is preserved by Darboux transformation. Now, suppose that $T \in \mathcal{E}(\mathfrak{L}_\lambda)$, $\mathrm{Sol}(\mathfrak{L}_\lambda)$ and $\mathrm{Sol}(\widetilde{\mathfrak{L}}_\lambda)$ the solutions space for $\mathfrak{L}_\lambda\Psi^{(-)} = 0$ and $\widetilde{\mathfrak{L}}_\lambda\Psi^{(-)} = 0$ respectively. To transform $\mathcal{E}(\mathfrak{L}_\lambda)$ into $\mathcal{E}(\widetilde{\mathfrak{L}}_\lambda)$ we follows the diagram:

$$\begin{tikzcd} \mathrm{Sol}(\mathfrak{L}_\lambda) \arrow[r, "T"] \arrow[d, leftarrow, "A^\dagger"'] & \mathrm{Sol}(\mathfrak{L}_\lambda) \arrow[d, "A"] \\ \mathrm{Sol}(\widetilde{\mathfrak{L}}_\lambda) \arrow[r, "\widetilde{T}"'] & \mathrm{Sol}(\widetilde{\mathfrak{L}}_\lambda) \end{tikzcd} \qquad \Rightarrow \widetilde{T} = ATA^\dagger \mathrm{mod}(\widetilde{\mathfrak{L}}_\lambda) \in \mathcal{E}(\widetilde{\mathfrak{L}}_\lambda),$$

where $A^\dagger$ and A are the raising and lowering operators. □

Example. Assume the Schrödinger equation $\mathcal{L}_\lambda$ with potential $V = V_- = 0$, which means that $\Lambda = \mathbb{C}$. If we choose $\lambda_1 = 0$ and as particular solution $\Psi_0^{(-)} = x$, then for $\lambda \neq 0$ the general solution is given by

$$\Psi_\lambda^{(-)} = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}.$$

Applying the Darboux transformation DT, we have $\mathrm{DT}(\mathcal{L}_\lambda) = \widetilde{\mathcal{L}}_\lambda$, where

$$\mathrm{DT}(V_-) = V_+ = \frac{2}{x^2}$$

and for $\lambda \neq 0$

$$\mathrm{DT}(\Psi_\lambda^{(-)}) = \Psi_\lambda^{(+)} = \frac{c_1(\sqrt{-\lambda}x - 1)e^{\sqrt{-\lambda}x}}{x} - \frac{c_2(\sqrt{-\lambda}x + 1)e^{-\sqrt{-\lambda}x}}{x}.$$

We can see that $\widetilde{K} = K = \mathbb{C}(x)$ for all $\lambda \in \Lambda$ and the Picard-Vessiot extensions are given by $L_0 = \widetilde{L}_0 = \mathbb{C}(x)$, $L_\lambda = \widetilde{L}_\lambda = \mathbb{C}(x, e^{\sqrt{\lambda}x})$ for $\lambda \in \mathbb{C}^*$. In this way, we have that $\mathrm{DGal}(L_0/K) = \mathrm{DGal}(\widetilde{L}_0/K) = e$; for $\lambda \neq 0$, we have $\mathrm{DGal}(L_\lambda/K) = \mathrm{DGal}(\widetilde{L}_\lambda/K) = \mathbb{G}_m$. The eigenrings of the operators $\mathfrak{L}_0$ and $\mathfrak{L}_\lambda$ are given by

$$\mathcal{E}(\mathfrak{L}_0) = \mathrm{Vect}\left(1, x\partial_x, x\partial_x - 1, x^2\partial_x - x\right),$$

$$\mathcal{E}(\widetilde{\mathfrak{L}}_0) = \mathrm{Vect}\left(1, x\partial_x - 1, x^4\partial_x - 2x^3, \frac{\partial_x}{x^2} + \frac{1}{x^3}\right)$$

and for $\lambda \neq 0$

$$\mathcal{E}(\mathfrak{L}_\lambda) = \mathrm{Vect}\left(1, \partial_x\right), \quad \mathcal{E}(\widetilde{\mathfrak{L}}_\lambda) = \mathrm{Vect}\left(1, -\left(\lambda + \frac{1}{x^2}\right)\partial_x - \frac{1}{x^3}\right),$$

where $\mathcal{L}_\lambda := \mathfrak{L}_\lambda \Psi^{(-)} = 0$ and $\widetilde{\mathcal{L}}_\lambda := \widetilde{\mathfrak{L}}_\lambda \Psi^{(+)} = 0$.

Applying iteratively the Darboux transformation, theorem 2.2.8, and by propositions 2.2.9, 2.2.10 we have the following results.

Proposition 2.2.11 (Galoisian version of DT_n)**.** *Let be $\Lambda \neq \emptyset$, $\mathcal{L}_\lambda^{(n)}$ given by $H^{(n)}\Psi^{(n)} = \lambda\Psi^{(n)}$, $V_n \in K_n$, $K_0 = K$, $V_0 = V_-$, $H^{(0)} = H_-$, $\Psi^{(0)} = \Psi^{(-)}$ and $L_\lambda^{(n)}$ the Picard-Vessiot extension of $\mathcal{L}_\lambda^{(n)}$. Let $\mathcal{L}_\lambda^{(n+1)}$ given by $H^{(n+1)}\Psi^{(n+1)} = \lambda\Psi^{(n+1)}$, $V_{n+1} \in K_{n+1}$. Let DT_n such that $\mathcal{L}_\lambda^{(n)} \mapsto \mathcal{L}_\lambda^{(n+1)}$, $V_n \mapsto V_{n+1}$, $\Psi_\lambda^{(n)} \mapsto \Psi_\lambda^{(n+1)}$ and $L_\lambda^{(n+1)}$ the Picard-Vessiot extension of $\mathcal{L}_\lambda^{(n+1)}$. Then the following statements holds:*

i) $\mathrm{DT}_n(V_-) = \mathrm{DT}(V_n) = V_{n+1} = V_n - 2\partial_x^2\left(\ln \Psi_\lambda^{(n)}\right) = V_- - 2\sum_{k=0}^n \partial_x^2\left(\ln \Psi_{\lambda_k}^{(k)}\right)$, *where $\Psi_{\lambda_k}^{(k)}$ is a particular solution for $\lambda = \lambda_k$, $k = 0, \ldots, n$. In particular, if $\lambda_n = \lambda_0$ and $\Lambda = \mathbb{C}$, then there exists $\Psi_{\lambda_n}^{(n)}$ such that $V_n \neq V_{n-2}$, with $n \geq 2$.*

ii) $\mathrm{DT}(\Psi_\lambda^{(n)}) = \mathrm{DT}_n(\Psi_\lambda^{(-)}) = \Psi_\lambda^{(n+1)} = \partial_x\Psi_\lambda^{(n)} - \Psi_\lambda^{(n)}\frac{\partial_x\Psi_{\lambda_n}^{(n)}}{\Psi_{\lambda_n}^{(n)}} = \frac{W(\Psi_{\lambda_n}^{(n)},\Psi_\lambda^{(n)})}{W(\Psi_{\lambda_n}^{(n)})}$ *where $\Psi_\lambda^{(n)}$ is a general solution for $\lambda \in \Lambda \setminus \{\lambda_n\}$ of $\mathcal{L}_\lambda^{(n)}$.*

iii) $K_{n+1} = K_n\left\langle \partial_x(\ln \Psi_{\lambda_n}^{(n)})\right\rangle$.

iv) DT_n *is isogaloisian and virtually strongly isogaloisian. Furthermore, if $\partial_x(\ln \Psi_{\lambda_n}^{(n)}) \in K_n$ then DT_n is strongly isogaloisian.*

v) *The eigenrings of $H^{(n)} - \lambda$ and $H^{(n+1)} - \lambda$ are isomorphic.*

Proof. By induction on theorem 2.2.8 we obtain i) and ii). By induction on proposition 2.2.9 we obtain iii) and iv). By induction on proposition 2.2.10 we obtain v).

□

Example. Starting with $V = 0$, the following potentials can be obtained using Darboux iteration DT_n (see [14, 17]).

$$I)\ V_n = \frac{n(n-1)b^2}{(bx+c)^2}, \quad II)\ V_n = \frac{m^2n(n-1)(b^2-a^2)}{(a\cosh(mx)+b\sinh(mx))^2},$$

$$III)\ V_n = \frac{-4abm^2n(n-1)}{(ae^{mx}+be^{-mx})^2}, \quad IV)\ V_n = \frac{m^2n(n-1)(b^2+a^2)}{(a\cos(mx)+b\sin(mx))^2}.$$

In particular for the rational potential given in I), we have $K = K_n = \mathbb{C}(x)$ and for $\lambda_n = \lambda = 0$, we have

$$\Psi_0^{(n)} = \frac{c_1}{(bx+c)^n} + c_2(bx+c)^{n+1}, \text{ so that } \mathrm{DGal}(L_0/K) = \mathrm{DGal}(L_0^{(n)}/K) = e,$$

$$\mathcal{E}(H^{(n)}) = \mathrm{Vect}\left(1, x\partial_x - 1, x^{2n+2}\partial_x - (n+1)x^{2n+1}, \frac{\partial_x}{x^{2n}} + \frac{n}{x^{2n+1}}\right),$$

whilst for $\lambda \neq 0$ and $\lambda_n = 0$, the general solution $\Psi_\lambda^{(n)}$ is given by

$$\Psi_\lambda^{(n)}(x) = c_1 f_n(x,\lambda) h_n(\sin(\sqrt{\lambda}x) + c_2 g_n(x,\lambda) j_n(\cos(\sqrt{\lambda}x),$$

where $f_n, g_n, h_n, j_n \in \mathbb{C}(x)$, so that

$$\mathrm{DGal}(L_\lambda/K) = \mathrm{DGal}(L_\lambda^{(n)}/K) = \mathbb{G}_m,$$

and

$$\dim_{\mathbb{C}} \mathcal{E}(H-\lambda) = \dim_{\mathbb{C}} \mathcal{E}(H^{(n)}-\lambda) = 2.$$

Proposition 2.2.12 (Galoisian version of CI_n)**.** *Consider $\mathcal{L}_\lambda$ given by $H\Psi = \lambda\Psi$, $H = -\partial_x^2 + V$, $V \in K$, such that* $\mathrm{Card}(\Lambda) > n$ *for a fixed $n \in \mathbb{Z}_+$. Let $\mathcal{L}_\lambda^{(n)}$ be given by $H^{(n)}\Psi^{(n)} = \lambda\Psi^{(n)}$, where $H^{(n)} = \partial_x^2 + V_n$, $V_n \in K_n$. Let CI_n be the transformation such that $\mathcal{L}_\lambda \mapsto \mathcal{L}_\lambda^{(n)}$, $V \mapsto V_n$, $(\Psi_{\lambda_1},\ldots,\Psi_{\lambda_n},\Psi_\lambda) \mapsto \Psi_\lambda^{(n)}$, where for $k = 1,\ldots,n$ and the equation $\mathcal{L}_\lambda$, the function Ψ_λ is the general solution for $\lambda \neq \lambda_k$ and Ψ_{λ_k} is a particular solution for $\lambda = \lambda_k$. Then the following statements holds:*

i) $\mathrm{CI}_n(\mathcal{L}_\lambda) = \mathcal{L}_\lambda^{(n)}$ *where* $\mathrm{CI}_n(V) = V_n = V - 2\partial_x^2\left(\ln W(\Psi_{\lambda_1},\ldots,\Psi_{\lambda_n})\right)$ *and*

$$\mathrm{CI}_n(\Psi_\lambda) = \Psi_\lambda^{(n)} = \frac{W(\Psi_{\lambda_1},\ldots,\Psi_{\lambda_n},\Psi_\lambda)}{W(\Psi_{\lambda_1},\ldots,\Psi_{\lambda_n})},$$

where $\Psi_\lambda^{(n)}$ is the general solution of $\mathcal{L}_\lambda^{(n)}$.

ii) $K_n = K\langle \partial_x\left(\ln W(\Psi_{\lambda_1},\ldots,\Psi_{\lambda_n})\right)\rangle$.

iii) CI_n *is isogaloisian and virtually strongly isogaloisian. Furthermore, if*

$$\partial_x\left(\ln W(\Psi_{\lambda_1},\ldots,\Psi_{\lambda_n})\right) \in K_n,$$

then CI_n *is strongly isogaloisian.*

iv) *The eigenrings of $H-\lambda$ and $H^{(n)}-\lambda$ are isomorphic.*

Proof. By induction on theorem 2.2.8 we obtain i). By induction on proposition 2.2.9 we obtain ii) and iii). By induction on proposition 2.2.10 we obtain iv). □

Example. To illustrate the Crum iteration with rational potentials, we consider $V = \frac{2}{x^2}$. The general solution of $\mathcal{L}_\lambda := H\Psi = \lambda\Psi$ is

$$\frac{c_1 e^{kx}(kx-1)}{x} + \frac{c_2 e^{-kx}(kx+1)}{x}, \quad \lambda = -k^2,$$

the eigenfunctions for $\lambda_1 = -1$, and $\lambda_2 = -4$, are respectively given by

$$\Psi_{-1} = \frac{e^{-x}(x+1)}{x}, \quad \Psi_{-4} = \frac{e^{-2x}(2x+1)}{2x}.$$

Thus, we obtain

$$\mathrm{CI}_2(V) = V_2 = \frac{8}{(2x+3)^2}$$

and the general solution of $\mathcal{L}_\lambda^{(2)} := H^{(2)}\Psi^{(2)} = \lambda\Psi^{(2)}$ is

$$\mathrm{CI}_2(\Psi_\lambda) = \Psi_\lambda^{(2)} = \frac{c_1\,(k\,(2\,x+3)-2)\,e^{kx}}{2\,x+3} + \frac{c_2\,(2+k\,(2\,x+3))\,e^{-kx}}{4\,x+6}, \quad \lambda = -k^2.$$

The differential Galois groups and eigenrings are given by:

$$\mathrm{DGal}(L_0/K) = \mathrm{DGal}(L_0^{(2)}/K) = e, \quad \dim_{\mathbb{C}} \mathcal{E}(H) = \dim_{\mathbb{C}} \mathcal{E}(H^{(2)}) = 4,$$

and for $\lambda \neq 0$

$$\mathrm{DGal}(L_\lambda/K) = \mathrm{DGal}(L_\lambda^{(2)}/K) = \mathbb{G}_m, \quad \dim_{\mathbb{C}} \mathcal{E}(H-\lambda) = \dim_{\mathbb{C}} \mathcal{E}(H^{(2)}-\lambda) = 2.$$

Proposition 2.2.13. *The supersymmetric partner potentials $V_\pm$ are rational functions if and only if the superpotential W is a rational function.*

Proof. The supersymmetric partner potentials $V_\pm$ are written as $V_\pm = W^2 \pm \partial_x W$. We start considering the superpotential $W \in \mathbb{C}(x)$, so trivially we have that $V_\pm \in \mathbb{C}(x)$. Now assuming that $V_\pm \in \mathbb{C}(x)$ we have that $\partial_x W \in \mathbb{C}(x)$ and $W^2 \in \mathbb{C}(x)$, which implies that $W\partial_x W \in \mathbb{C}(x)$ and therefore $W \in \mathbb{C}(x)$.

□

Corollary 2.2.14. *The superpotential $W \in \mathbb{C}(x)$ if and only if* DT *is strong isogaloisian.*

Proof. Assume that the superpotential $W \in \mathbb{C}(x)$. Thus, by proposition 2.2.9, DT is strong isogaloisian. Now, assume that DT is strong isogaloisian. Thus, $V_\pm \in \mathbb{C}(x)$ and by proposition 2.2.13 we have that $W \in \mathbb{C}(x)$.

□

The following definition is a partial Galoisian adaptation of the original definition given in [37] ($K = \mathbb{C}(x)$). The complete Galoisian adaptation is given when K is any differential field.

Definition 2.2.15 (Rational Shape Invariant Potentials)**.** Assume $V_\pm(x;\mu) \in \mathbb{C}(x;\mu)$, where μ is a family of parameters. The potential $V = V_- \in \mathbb{C}(x)$ is said to be rational shape invariant potential with respect to μ and $E = E_n$ being $n \in \mathbb{Z}_+$, if there exists f such that

$$V_+(x;a_0) = V_-(x;a_1) + R(a_1), \quad a_1 = f(a_0), \quad E_n = \sum_{k=2}^{n+1} R(a_k), \quad E_0 = 0.$$

Remark 2.2.16. We propose the following steps to check whether $V \in \mathbb{C}(x)$ is shape invariant.

Step 1. Introduce parameters in $W(x)$ to obtain $W(x;\mu)$, write $V_\pm(x;\mu) = W^2(x;\mu) \pm \partial_x W(x;\mu)$, and replace μ by a_0 and a_1.

Step 2. Obtain polynomials $\mathcal{P} \in \mathbb{C}[x;a_0,a_1]$ and $\mathcal{Q} \in \mathbb{C}[x;a_0,a_1]$ such that

$$\partial_x(V_+(x;a_0) - V_-(x;a_1)) = \frac{\mathcal{P}(x;a_0,a_1)}{\mathcal{Q}(x;a_0,a_1)}.$$

Step 3. Set $\mathcal{P}(x;a_0,a_1) \equiv 0$, as polynomial in x, to obtain a_1 in function of a_0, i.e., $a_1 = f(a_0)$. Also obtain $R(a_1) = V_+(x;a_0) - V_-(x;a_1)$ and verify that exists $k \in \mathbb{Z}^+$ such that $R(a_1) + \cdots + R(a_k) \neq 0$.

Example. Consider the superpotential of the three dimensional harmonic oscillator $W(r;\ell) = r - \frac{\ell+1}{r}$. By step 1, the supersymmetric partner potentials are

$$V_-(r;\ell) = r^2 + \frac{\ell(\ell+1)}{r^2} - 2\ell - 3, \quad V_+(r;\ell) = r^2 + \frac{(\ell+1)(\ell+2)}{r^2} - 2\ell - 1.$$

By step 2, we have $\partial_r(V_+(r;a_0) - V_-(r;a_1)) = -2\frac{a_0^2+3a_0-a_1^2-a_1+2}{r^3}$. By step 3, $(a_0+1)(a_0+2) = a_1(a_1+1)$, so that $a_1 = f(a_0) = a_0+1$, $a_n = f(a_{n-1}) = a_0+n$, $R(a_1) = 2$. Thus, we obtain the energy levels $E_n = 2n$ and the wave functions $\Psi_n^{(-)}(r;\ell) = A^\dagger(r;\ell)\cdots A^\dagger(r;\ell+n-1)\Psi_0^{(-)}(r;\ell+n)$, compare with [31].

By theorem 2.2.8 and propositions 2.2.9, 2.2.10 and 2.2.13 we have the following result.

Theorem 2.2.17. *Consider $\mathcal{L}_n := H\Psi^{(-)} = E_n\Psi^{(-)}$ with Picard-Vessiot extension L_n, where $n \in \mathbb{Z}_+$. If $V = V_- \in \mathbb{C}(x)$ is a shape invariant potential with respect to $E = E_n$, then*

$$\mathrm{DGal}(L_{n+1}/K) = \mathrm{DGal}(L_n/K), \quad \mathcal{E}(H - E_{n+1}) \simeq \mathcal{E}(H - E_n), \quad n > 0.$$

Remark 2.2.18. The differential automorphisms σ commutes with the raising and lowering operators A and $A^\dagger$ due to $W \in \mathbb{C}(x)$. Furthermore the wave functions $\Psi_n^{(-)}$ can be written as $\Psi_n^{(-)} = P_n f_n \Psi_0^{(-)}$, where P_n is a polynomial of degree n in x and f_n is a sequence of functions being $f_0(x) = 1$ such as was shown in the case of Harmonic oscillators and Coulomb potentials.

2.3 The Role of the Algebrization in Supersymmetric Quantum Mechanics

In supersymmetric quantum mechanics, there exists potentials which are not rational functions and, for this reason, it is difficult to apply our Galoisian approach such as in section 2.2. In this section we give a solution to this problem presenting some results concerning differential equations with non-rational coefficients. For these differential equations it is useful, when is possible, to replace it by a new differential equation over the Riemann sphere $\mathbb{P}^1$ (that is, with rational coefficients). To do this, we can use a change of variables. The equation over $\mathbb{P}^1$ is called the *algebraic form* or *algebrization* of the original equation.

This algebraic form dates back to the 19th century (Liouville, Darboux), but the problem of obtaining the algebraic form (if it exists) of a given differential equation is in general not an easy task. Here we develop a new method using the concept of *Hamiltonian change of variables*. This change of variables allow us to compute the algebraic form of a large number of differential equations of different types. In particular, for second order linear differential equations, we can apply *Kovacic's algorithm* over the algebraic form to solve the original equation.

The following definition can be found in [12, 47, 48].

Definition 2.3.1 (Pullbacks of differential equations)**.** Let $\mathfrak{L}_1 \in K_1[\partial_z]$ and $\mathfrak{L}_2 \in K_2[\partial_x]$ be differential operators, the expression $\mathfrak{L}_2 \otimes (\partial_x + v)$ refers to the operator whose solutions are the solutions of $\mathfrak{L}_2$ multiplied by the solution $e^{-\int v dx}$ of $\partial_x + v$.

- $\mathfrak{L}_2$ is a *proper pullback* of $\mathfrak{L}_1$ by means of $f \in K_2$ if the change of variable $z = f(x)$ changes $\mathfrak{L}_1$ into $\mathfrak{L}_2$.
- $\mathfrak{L}_2$ is a *pullback* (also known as weak pullback) of $\mathfrak{L}_1$ by means of $f \in K_2$ if there exists $v \in K_2$ such that $\mathfrak{L}_2 \otimes (\partial_x + v)$ is a proper pullback of $\mathfrak{L}_1$ by means of f.

In case of compact Riemann surfaces, the geometric mechanism behind the algebrization is a ramified covering of compact Riemann surfaces, see [68, 67].

2.3.1 Second Order Linear Differential Equations

Some results presented in this subsection also can be found in [5, §2].

Proposition 2.3.2 (Change of the independent variable, [5])**.** *Let us consider the following equation, with coefficients in $\mathbb{C}(z)$:*

$$\mathcal{L}_z := \partial_z^2 y + a(z)\partial_z y + b(z)y = 0, \tag{2.4}$$

and $\mathbb{C}(z) \hookrightarrow L$ the corresponding Picard-Vessiot extension. Let (K, δ) be a differential field with $\mathbb{C}$ as field of constants. Let $\theta \in K$ be a non-constant element.

Then, by the change of variable $z=\theta(x)$, equation (2.4) is transformed in

$$\mathcal{L}_x := \partial_x^2 r + \left(a(\theta)\partial_x\theta - \frac{\partial_x^2\theta}{\partial_x\theta} \right)\partial_x r + b(\theta)(\partial_x\theta)^2 r = 0, \quad \partial_x = \delta, \quad r = y\circ\theta. \tag{2.5}$$

Let $K_0 \subset K$ be the smallest differential field containing θ and $\mathbb{C}$. Then equation (2.5) is a differential equation with coefficients in K_0. Let $K_0 \hookrightarrow L_0$ be the corresponding Picard-Vessiot extension. Assume that

$$\mathbb{C}(z) \to K_0, \quad z \mapsto \theta$$

is an algebraic extension, then

$$(\mathrm{DGal}(L_0/K_0))^0 = (\mathrm{DGal}(L/\mathbb{C}(z)))^0.$$

Proposition 2.3.3. *Assume $\mathcal{L}_x$ and $\mathcal{L}_z$ as in proposition 2.3.2. Let φ be the transformation given by*

$$\varphi : \begin{array}{l} z \mapsto \theta(x) \\ \\ \partial_z \mapsto \frac{1}{\partial_x\theta}\delta. \end{array}$$

Then $\mathrm{DGal}(L_0/K_0) \simeq \mathrm{DGal}(L/K_0\cap L) \subset \mathrm{DGal}(L/\mathbb{C}(z))$. Furthermore, if $K_0\cap L$ is algebraic over $\mathbb{C}(z)$, then $(\mathrm{DGal}(L_0/K_0))^0 \simeq (\mathrm{DGal}(L/\mathbb{C}(x)))^0$.

Proof. By Proposition 2.3.2, the transformation φ leads us to

$$\mathbb{C}(z) \simeq \varphi(\mathbb{C}(z)) \hookrightarrow K_0,$$

that is, we identify $\mathbb{C}(z)$ with $\varphi(\mathbb{C}(z))$, and so that we can view $\mathbb{C}(z)$ as a subfield of K_0 and then by the Kaplansky's diagram (see [53, 106]),

$$\begin{tikzcd}
 & L_0 \arrow[dl, dash] \arrow[dr, dash] \arrow[dr, dotted, shift left=2, "\mathrm{DGal}(L_0/K_0)"] & \\
L \arrow[dr, dash] \arrow[dr, dotted, shift right=2, "\mathrm{DGal}(L/(L\bigcap K_0))"'] & & K_0 \arrow[dl, dash] \\
 & L\bigcap K_0 \arrow[d, dash] & \\
 & \mathbb{C}(z) &
\end{tikzcd}$$

so that we have

$$\mathrm{DGal}(L_0/K_0) \simeq \mathrm{DGal}(L/K_0\cap L) \subset \mathrm{DGal}(L/\mathbb{C}(z))$$

and if $K_0\cap L$ is algebraic over $\mathbb{C}(z)$, then

$$(\mathrm{DGal}(L_0/K_0))^0 \simeq (\mathrm{DGal}(L/\mathbb{C}(z)))^0.$$

$\square$

Along the rest of this section we write $z = z(x)$ instead of θ.

Remark 2.3.4 (Hard Algebrization). The proper pullback from equation (2.5) to equation (2.4) is an algebrization process. Therefore, we can try to algebrize any second order linear differential equations with non-rational coefficients (proper pullback) if we can put it in the form of equation (2.5). To do this, which will be called *hard algebrization*, we use the following steps.

Step 1. Find $(\partial_x z)^2$ in the coefficient of y to obtain $\partial_x z$ and z.

Step 2. Divide by $(\partial_x z)^2$ to the coefficient of y to obtain $b(z)$ and check whether $b \in \mathbb{C}(z)$.

Step 3. Add $(\partial_x^2 z)/\partial_x z$ and divide by $\partial_x z$ to the coefficient of $\partial_x y$ to obtain $a(z)$ and check whether $a \in \mathbb{C}(z)$.

To illustrate this method, hard algebrization, we present the following example.

Example. In [92, p. 256], Singer presents the second order linear differential equation

$$\partial_x^2 r - \frac{1}{x(\ln x + 1)}\partial_x r - (\ln x + 1)^2 r = 0.$$

To algebrize this differential equation we choose $(\partial_x z)^2 = (\ln x + 1)^2$, so that $\partial_x z = \ln x + 1$ and for instance

$$z = \int (\ln x + 1)dx = x \ln x, \quad b(z) = -1.$$

Now we find $a(z)$ in the expression

$$a(z)(\ln x + 1) - \frac{1}{x(\ln x + 1)} = -\frac{1}{x(\ln x + 1)},$$

obtaining $a(z) = 0$. So that the new differential equation is given by $\partial_z^2 y - y = 0$, in which $y(z(x)) = r(x)$ and one basis of solutions of this differential equation is given by $\langle e^z, e^{-z}\rangle$. Thus, the respective basis of solutions of the first differential equation is given by $\langle e^{x \ln x}, e^{-x \ln x}\rangle$.

In general, this method is not clear because the quest of $z = z(x)$ in $b(z)(\partial_x z)^2$ can be purely a lottery, or simply there is not exists z such that $a(z), b(z) \in \mathbb{C}(z)$. For example, the equations presented by Singer in [92, p. 257, 261, 270] and given by

$$\partial_x^2 r + \frac{\mp 2x \ln^2 x \mp 2x \ln x - 1}{x \ln x + x}\partial_x r + \frac{-2x \ln^2 x - 3x \ln x - x \mp 1}{x \ln x + x} r = 0,$$

$$\partial_x^2 r + \frac{4x \ln x + 2x}{4x^2 \ln x}\partial_x r - \frac{1}{4x^2 \ln x} r = 0, \quad (x^2 \ln^2 x)\partial_x^2 r + (x \ln^2 x - 3x \ln x)\partial_x r + 3r = 0,$$

cannot be algebrized systematically with this method, although it corresponds to pullbacks (not proper pullback) of differential equations with constant coefficients.

In [21], Bronstein and Fredet developed and implemented an algorithm to solve differential equations over $\mathbb{C}(x, e^{\int f})$ without algebrizing it, see also [35]. As an application of proposition 2.3.2 we have the following result[1].

Proposition 2.3.5 (Linear differential equation over $\mathbb{C}(x, e^{\int f})$, [5]). *Let $f \in \mathbb{C}(x)$ be a rational function. Then, the differential equation*

$$\partial_x^2 r - \left(f + \frac{\partial_x f}{f} - f e^{\int f} a\left(e^{\int f}\right)\right)\partial_x r + \left(f\left(e^{\int f}\right)\right)^2 b\left(e^{\int f}\right) r = 0, \tag{2.6}$$

is algebrizable by the change $z = e^{\int f}$ and its algebraic form is given by

$$\partial_z^2 y + a(z)\partial_z y + b(z) y = 0, \quad r(x) = y(z(x)).$$

Proof. Assume that $r(x) = y(z(x))$, and $z = z(x) = e^{\int f dx}$. We can see that

$$\partial_x z = f z, \quad \partial_z y = \frac{\partial_x r}{f z}, \quad \partial_z^2 y = \frac{1}{f z}\partial_x\left(\frac{\partial_x r}{f z}\right) = \frac{1}{\left(f e^{\int f}\right)^2}\left(\partial_x^2 r - f + \left(\frac{\partial_x f}{f}\right)\right)\partial_x r,$$

replacing in $\partial_z^2 y + a(z)\partial_z y + b(z) y = 0$ we obtain equation (2.6). □

Example. The differential equation

$$\partial_x^2 r - \left(x + \frac{1}{x} - 2x e^{x^2}\right)\partial_x r + \lambda x^2 e^{x^2} r = 0,$$

is algebrizable by the change $z = e^{\frac{x^2}{2}}$ and its algebraic form is given by

$$\partial_z^2 y + 2z\partial_z y + \lambda y = 0.$$

Remark 2.3.6. According to proposition 2.3.5, we have the following cases.

1. $f = n\frac{\partial_x h}{h}$, for a rational function h, $n \in \mathbb{Z}_+$, we have the trivial case, both equations are over the Riemann sphere and they have the same differential field, so that does not need to be algebrized.

2. $f = \frac{1}{n}\frac{\partial_x h}{h}$, for a rational function h, $n \in \mathbb{Z}^+$, (2.6) is defined over an algebraic extension of $\mathbb{C}(x)$ and so that this equation is not necessarily over the Riemann sphere.

[1] This result is given in [5, §2], but we include here the proof for completeness.

3. $f \neq q\frac{\partial_x h}{h}$, for any rational function h, $q \in \mathbb{Q}$, (2.6) is defined over a transcendental extension of $\mathbb{C}(x)$ and so that this equation is not over the Riemann sphere.

To algebrize second order linear differential equations is easier when the term in $\partial_x r$ is absent, that is, in the form of equation (1.2) and the change of variable is *Hamiltonian*.

Definition 2.3.7 (Hamiltonian change of variable, [5])**.** A change of variable $z = z(x)$ is called *Hamiltonian* if $(z(x), \partial_x z(x))$ is a solution curve of the autonomous classical one degree of freedom Hamiltonian system

$$\begin{array}{l} \partial_x z = \partial_w H \\ \partial_x w = -\partial_z H \end{array} \quad \text{with} \quad H = H(z,w) = \frac{w^2}{2} + V(z),$$

for some $V \in K$.

Remark 2.3.8. Assume that we algebrize equation (2.5) through a Hamiltonian change of variables $z = z(x)$, i.e., $V \in \mathbb{C}(z)$. Then, $K_0 = \mathbb{C}(z, \partial_x z, \ldots)$, but, we have the algebraic relation,

$$(\partial_x z)^2 = 2h - 2V(z), \quad h = H(z, \partial_x z) \in \mathbb{C},$$

so that $K_0 = \mathbb{C}(z, \partial_x z)$ is an algebraic extension of $\mathbb{C}(z)$. By proposition 2.3.2 the identity connected component of the differential Galois group is conserved. On the other hand, we can identify a Hamiltonian change of variable $z = z(x)$ when there exists $\alpha \in K$ such that $(\partial_x z)^2 = \alpha(z)$. Thus, we introduce the *Hamiltonian algebrization*, which corresponds to the algebrization process done through a Hamiltonian change of variable.

The following result, which also can be found in [5, §2], is an example of Hamiltonian algebrization and correspond to the case of reduced second order linear differential equations.

Proposition 2.3.9 (Hamiltonian Algebrization, [5])**.** *The differential equation*

$$\partial_x^2 r = q(x) r$$

is algebrizable through a Hamiltonian change of variable $z = z(x)$ if and only if there exist f, α such that

$$\frac{\partial_z \alpha}{\alpha}, \quad \frac{f}{\alpha} \in \mathbb{C}(z), \ \textit{where } f(z(x)) = q(x), \quad \alpha(z) = 2(H - V(z)) = (\partial_x z)^2.$$

Furthermore, the algebraic form of the equation $\partial_x^2 r = q(x) r$ is

$$\partial_z^2 y + \frac{1}{2}\frac{\partial_z \alpha}{\alpha}\partial_z y - \frac{f}{\alpha} y = 0, \quad r(x) = y(z(x)). \tag{2.7}$$

Remark 2.3.10 (Using the Algebrization Method). The goal is to algebrize the differential equation $\partial_x^2 r = q(x)r$, so that we propose the following steps.

Step 1 Find a *Hamiltonian change of variable* $z = z(x)$ and two functions f and α such that $q(x) = f(z(x))$ and $(\partial_x z(x))^2 = \alpha(z(x))$.

Step 2 Verify whether or not $f(z)/\alpha(z) \in \mathbb{C}(z)$ and $\partial_z \alpha(z)/\alpha(z) \in \mathbb{C}(z)$ to see if the equation $\partial_x^2 r = q(x)r$ is algebrizable.

Step 3 If the equation $\partial_x^2 r = q(x)r$ is algebrizable, its algebrization is

$$\partial_z^2 y + \frac{1}{2}\frac{\partial_z \alpha}{\alpha}\partial_z y - \frac{f}{\alpha}y = 0, \quad y(z(x)) = r(x).$$

When we have algebrized the differential equation $\partial_x^2 r = q(x)r$, we study its integrability, Eigenring and its differential Galois group.

Examples. Consider the following examples.

- Given the differential equation $\partial_x^2 r = f(\tan x)r$ with $f \in \mathbb{C}(\tan x)$, we can choose $z = z(x) = \tan x$ to obtain $\alpha(z) = (1+z^2)^2$, so that $z = z(x)$ is a Hamiltonian change of variable. We can see that $\frac{\partial_z \alpha}{\alpha}, \frac{f}{\alpha} \in \mathbb{C}(z)$ and the algebraic form of the differential equation $\partial_x^2 r = f(\tan x)r$ with this Hamiltonian change of variable is

$$\partial_z^2 y + \frac{2z}{1+z^2}\partial_z y - \frac{f(z)}{(1+z^2)^2}y = 0, \quad y(\tan x) = r(x).$$

- Given the differential equation

$$\partial_x^2 r = \frac{\sqrt{1+x^2}+x^2}{1+x^2}r,$$

we can choose $z = z(x) = \sqrt{1+x^2}$ to obtain

$$f(z) = \frac{z^2+z-1}{z^2}, \quad \alpha(z) = \frac{z^2-1}{z^2},$$

so that $z = z(x)$ is a Hamiltonian change of variable. We can see that $\frac{\partial_z \alpha}{\alpha}, \frac{f}{\alpha} \in \mathbb{C}(z)$ and the algebraic form for this case is

$$\partial_z^2 y + \frac{1}{z(z^2-1)}\partial_z y - \frac{z^2+z-1}{z^2-1}y = 0, \quad y(\sqrt{1+x^2}) = r(x).$$

We remark that in general the method of Hamiltonian algebrization is not an algorithm, because the problem is to obtain a suitable Hamiltonian H satisfying definition 2.3.7. We present now a particular case of Hamiltonian algebrization considered as an algorithm[2].

[2] Proposition 2.3.11 is a slight improvement of a similar result given in [5, §2]. Furthermore, we include the proof here for completeness.

Proposition 2.3.11 (Hamiltonian Algebrization Algorithm, [5]). *Consider $q(x) = g(z_1, \cdots, z_n)$, where $z_i = e^{\lambda_i x}$, $\lambda_i \in \mathbb{C}^*$. The equation $\partial_x^2 r = q(x)r$ is algebrizable if and only if.*

$$\frac{\lambda_i}{\lambda_j} \in \mathbb{Q}^*, \quad 1 \leq i \leq n, 1 \leq j \leq n, \quad g \in \mathbb{C}(z).$$

Furthermore, $\lambda_i = c_i\lambda$, where $\lambda \in \mathbb{C}^*$ and $c_i \in \mathbb{Q}^*$ and for the Hamiltonian change of variable

$$z = e^{\frac{\lambda x}{q}}, \text{ where } c_i = \frac{p_i}{q_i}, p_i, q_i \in \mathbb{Z}^*, \ \gcd(p_i, q_i) = 1 \text{ and } q = \operatorname{lcm}(q_1, \cdots, q_n),$$

the algebrization of the differential equation $\partial_x^2 r = q(x)r$ is

$$\partial_z^2 y + \frac{1}{z}\partial_z y - q^2 \frac{g(z^{m_1}, \ldots, z^{m_n})}{\lambda^2 z^2} y = 0, \quad m_i = \frac{qp_i}{q_i}, \quad y(z(x)) = r(x).$$

Proof. Assuming $\lambda_i/\lambda_j = c_{ij} \in \mathbb{Q}^*$ we can see that there exists $\lambda \in \mathbb{C}^*$ and $c_i \in \mathbb{Q}^*$ such that $\lambda_i = \lambda c_i$, so that

$$e^{\lambda_i x} = e^{c_i \lambda x} = e^{\frac{p_i}{q_i}\lambda x} = \left(e^{\frac{\lambda}{q}x}\right)^{\frac{qp_i}{q_i}}, \ p_i, q_i \in \mathbb{Z}^*, \ \gcd(p_i, q_i) = 1, \ \operatorname{lcm}(q_1, \ldots, q_n) = q.$$

Now, setting $z = z(x) = e^{\frac{\lambda}{q}x}$ we can see that

$$f(z) = g(z^{m_1}, \ldots, z^{m_n}), \quad m_i = \frac{qp_i}{q_i}, \quad \alpha = \frac{\lambda^2 z^2}{q^2}.$$

Due to $q|q_i$, we have that $m_i \in \mathbb{Z}$, so that

$$\frac{\partial_z \alpha}{\alpha}, \quad \frac{f}{\alpha} \in \mathbb{C}(z)$$

and the algebraic form is given by

$$\partial_z^2 y + \frac{1}{z}\partial_z y - q^2 \frac{g(z^{m_1}, \ldots, z^{m_n})}{\lambda^2 z^2} y = 0, \quad y(z(x)) = r(x).$$

□

Remark 2.3.12. Propositions 2.3.9 and 2.3.11 allow the algebrization of a large number of second order differential equations, see for example [77]. In particular, under the assumptions of proposition 2.3.11, we can algebrize automatically differential equations with trigonometrical or hyperbolic coefficients.

Examples. Consider the following examples.

- Given the differential equation

$$\partial_x^2 r = \frac{e^{\frac{1}{2}x} + 3e^{-\frac{2}{3}x} - 2e^{\frac{5}{4}x}}{e^x + e^{-\frac{3}{2}x}} r,\ \lambda_1 = \frac{1}{2},\ \lambda_2 = -\frac{2}{3},\ \lambda_3 = \frac{5}{4},\ \lambda_4 = 1,\ \lambda_5 = -\frac{3}{2},$$

we see that $\lambda_i/\lambda_j \in \mathbb{Q}$, $\lambda = 1$, $q = \mathrm{lcm}(1, 2, 3, 4) = 12$ and the Hamiltonian change of variable for this case is $z = z(x) = e^{\frac{1}{12}x}$. We can see that

$$\alpha(z) = \frac{1}{144}z^2, \quad f(z) = \frac{z^6 + 3z^{-8} - 2z^{15}}{z^{12} + z^{-18}}, \quad \frac{\partial_z \alpha}{\alpha}, \frac{f}{\alpha} \in \mathbb{C}(z)$$

and the algebraic form is given by

$$\partial_z^2 y + \frac{1}{z}\partial_z y - 144\frac{z^6 + 3z^{-8} - 2z^{15}}{z^{14} + z^{-16}} y = 0, \quad y(e^{\frac{1}{12}x}) = r(x).$$

- Given the differential equation

$$\partial_x^2 r = (e^{2\sqrt{2}x} + e^{-\sqrt{2}x} - e^{3x})r,\ \lambda_1 = 2\sqrt{2},\ \lambda_2 = -\sqrt{2},\ \lambda_3 = 3,$$

we see that $\lambda_1/\lambda_2 \in \mathbb{Q}$, but $\lambda_1/\lambda_3 \notin \mathbb{Q}$, so that this differential equation cannot be algebrized.

We remark that it is possible to use the algebrization method to transform differential equations, although either the starting equation has rational coefficients or the transformed equation has not rational coefficients.

Examples. As illustration we present the following examples.

- Consider the following differential equation

$$\partial_x^2 r = \frac{x^4 + 3x^2 - 5}{x^2 + 1} r = 0,$$

we can choose $z = z(x) = x^2$ so that $\alpha = 4z$ and the new differential equation is

$$\partial_z^2 y + \frac{1}{2z}\partial_z y - \frac{z^2 + 2z - 5}{4z(z+1)} y = 0, \quad y(x^2) = r(x).$$

- Consider the Mathieu's differential equation $\partial_x^2 r = (a + b\cos(x))r$, we can choose $z(x) = \ln(\cos(x))$ so that $\alpha = e^{-2z} - 1$ and the new differential equation is

$$\partial_z^2 y - \frac{1}{1 - e^{2z}}\partial_z y - \frac{ae^{2z} + be^{3z}}{1 - e^{2z}} y = 0, \quad y(\ln(\cos(x))) = r(x).$$

Recently, the Hamiltonian algebrization (propositions 2.3.9 and 2.3.11) has been applied in [2, 3, 5] to obtain non-integrability in the framework of *Morales-Ramis theory* [68, 67].

2.3.2 The Operator $\widehat{\partial}_z$ and the Hamiltonian Algebrization

The generalization of proposition 2.3.2 to higher order linear differential equations is difficult. But, it is possible to obtain generalizations of proposition 2.3.9 by means of Hamiltonian change of variable. We recall that $z = z(x)$ is a Hamiltonian change of variable if there exists α such that $(\partial_x z)^2 = \alpha(z)$. More specifically, if $z = z(x)$ is a Hamiltonian change of variable, we can write $\partial_x z = \sqrt{\alpha}$, which leads us to the following notation: $\widehat{\partial}_z = \sqrt{\alpha}\partial_z$.

We can see that $\widehat{\partial}_z$ is a *derivation* because satisfy $\widehat{\partial}_z(f+g) = \widehat{\partial}_z f + \widehat{\partial}_z g$ and the Leibnitz rules

$$\widehat{\partial}_z(f \cdot g) = \widehat{\partial}_z f \cdot g + f \cdot \widehat{\partial}_z g, \quad \widehat{\partial}_z\left(\frac{f}{g}\right) = \frac{\widehat{\partial}_z f \cdot g - f \cdot \widehat{\partial}_z g}{g^2}.$$

We can notice that the chain rule is given by $\widehat{\partial}_z(f \circ g) = \partial_g f \circ g \widehat{\partial}_z(g) \neq \widehat{\partial}_g f \circ g \widehat{\partial}_z(g)$. The iteration of $\widehat{\partial}_z$ is given by

$$\widehat{\partial}_z^0 = 1, \quad \widehat{\partial}_z = \sqrt{\alpha}\partial_z, \quad \widehat{\partial}_z^n = \sqrt{\alpha}\partial_z\widehat{\partial}_z^{n-1} = \underbrace{\sqrt{\alpha}\partial_z\left(\dots\left(\sqrt{\alpha}\partial_z\right)\right)}_{n \text{ times } \sqrt{\alpha}\partial_z}.$$

We say that a Hamiltonian change of variable is rational when the potential $V \in \mathbb{C}(x)$ and for instance $\alpha \in \mathbb{C}(x)$. Along the rest of this thesis, we understand $\widehat{\partial}_z = \sqrt{\alpha}\partial_z$ where $z = z(x)$ is a Hamiltonian change of variable and $\partial_x z = \sqrt{\alpha}$. In particular, $\widehat{\partial}_z = \partial_z = \partial_x$ if and only if $\sqrt{\alpha} = 1$, i.e., $z = x$.

Theorem 2.3.13. *Consider the systems of linear differential equations* $[A]$ *and* $[\widehat{A}]$ *given respectively by*

$$\partial_x \mathbf{Y} = -A\mathbf{Y}, \quad \widehat{\partial}_z \widehat{\mathbf{Y}} = -\widehat{A}\widehat{\mathbf{Y}}, \quad A = [a_{ij}], \quad \widehat{A} = [\widehat{a}_{ij}], \quad \mathbf{Y} = [y_{i1}], \quad \widehat{\mathbf{Y}} = [\widehat{y}_{i1}],$$

where $a_{ij} \in K = \mathbb{C}(z(x), \partial_x(z(x)))$, $\widehat{a}_{ij} \in \mathbb{C}(z) \subseteq \widehat{K} = \mathbb{C}(z, \sqrt{\alpha})$, $1 \leq i \leq n$, $1 \leq j \leq n$, $a_{ij}(x) = \widehat{a}_{ij}(z(x))$ *and* $y_{i1}(x) = y_{i1}(z(x))$. *Suppose that* L *and* $\widehat{L}$ *are the Picard-Vessiot extensions of* $[A]$ *and* $[\widehat{A}]$ *respectively. If the transformation* φ *is given by*

$$\varphi: \begin{array}{l} x \mapsto z \\ a_{ij} \mapsto \widehat{a}_{ij} \\ y_{i1}(x) \mapsto \widehat{y}_{i1}(z(x)) \\ \partial_x \mapsto \widehat{\partial}_z \end{array},$$

then the following statements hold.

- $K \simeq \widehat{K}$, $(K, \partial_x) \simeq (\widehat{K}, \widehat{\partial}_z)$.
- $\mathrm{DGal}(L/K) \simeq \mathrm{DGal}(\widehat{L}/\widehat{K}) \subset \mathrm{DGal}(\widehat{L}/\mathbb{C}(z))$.
- $(\mathrm{DGal}(L/K))^0 \simeq (\mathrm{DGal}(\widehat{L}/\mathbb{C}(z))^0$.

- $\mathcal{E}(A) \simeq \mathcal{E}(\widehat{A})$.

Proof. We proceed as in the proof of proposition 2.3.3. As $z = z(x)$ is a rational Hamiltonian change of variable, the transformation φ leads us to

$$\mathbb{C}(z) \simeq \varphi(\mathbb{C}(z)) \hookrightarrow K, \quad K \simeq \widehat{K}, \quad \mathbb{C}(z) \hookrightarrow \widehat{K}, \quad (K, \partial_x) \simeq (\widehat{K}, \widehat{\partial}_z)$$

that is, we identify $\mathbb{C}(z)$ with $\varphi(\mathbb{C}(z))$, and so that we can view $\mathbb{C}(z)$ as a subfield of K and then, by the Kaplansky's diagram (see [53, 106]),

$$\begin{tikzcd}
& L \arrow[dl, dash] \arrow[dr, dash] \arrow[dr, dotted, bend left, "\mathrm{DGal}(L/K)"] & \\
\widehat{L} \arrow[dr, dash] \arrow[dr, dotted, bend right, "\mathrm{DGal}(\widehat{L}/\widehat{K})"'] & & K \arrow[dl, dash] \\
& \widehat{K} \arrow[d, dash] & \\
& \mathbb{C}(z) &
\end{tikzcd}$$

so that we have $\mathrm{DGal}(L/K) \simeq \mathrm{DGal}(\widehat{L}/\widehat{K}) \subset \mathrm{DGal}(\widehat{L}/\mathbb{C}(z))$, $(\mathrm{DGal}(L/K))^0 \simeq (\mathrm{DGal}(\widehat{L}/\mathbb{C}(z)))^0$, and $\mathcal{E}(A) \simeq \mathcal{E}(\widehat{A})$. $\square$

We remark that the transformation φ, given in theorem 2.3.13, is virtually strong isogaloisian when $\sqrt{\alpha} \notin \mathbb{C}(z)$ and for $\sqrt{\alpha} \in \mathbb{C}(z)$, φ is strong isogaloisian. Furthermore, by cyclic vector method (see [81]), we can write the systems $[A]$ and $[\widehat{A}]$ in terms of the differential equations $\mathcal{L}$ and $\widehat{\mathcal{L}}$. Thus, $\widehat{\mathcal{L}}$ is the proper pullback of $\mathcal{L}$ and $\mathcal{E}(\mathfrak{L}) \simeq \mathcal{E}(\widehat{\mathfrak{L}})$.

Example. Consider the system

$$[A] := \begin{array}{l} \partial_x\gamma_1 = -\frac{2\sqrt{2}}{e^x+e^{-x}}\gamma_3, \\[2ex] \partial_x\gamma_2 = \frac{e^x-e^{-x}}{e^x+e^{-x}}\gamma_3, \\[2ex] \partial_x\gamma_3 = \frac{2\sqrt{2}}{e^x+e^{-x}}\gamma_1 - \frac{e^x-e^{-x}}{e^x+e^{-x}}\gamma_2, \end{array}$$

which through the Hamiltonian change of variable $z = e^x$, and for instance $\sqrt{\alpha} = z$, it is transformed in the system

$$[\widehat{A}] := \begin{array}{l} \partial_z\widehat{\gamma}_1 = -\frac{2\sqrt{2}}{z^2+1}\widehat{\gamma}_3, \\[2ex] \partial_z\widehat{\gamma}_2 = \frac{z^2-1}{z(z^2+1)}\widehat{\gamma}_3, \\[2ex] \partial_z\widehat{\gamma}_3 = \frac{2\sqrt{2}}{z^2+1}\widehat{\gamma}_1 - \frac{z^2-1}{x(x^2+1)}\widehat{\gamma}_2. \end{array}$$

One solution of the system $[\widehat{A}]$ is given by

$$\frac{1}{z^2+1}\begin{pmatrix}\frac{\sqrt{2}}{2}(1-z^2)\\ z\\ -z\end{pmatrix},$$

and for instance,

$$\frac{1}{e^{2x}+1}\begin{pmatrix}\frac{\sqrt{2}}{2}(1-e^{2x})\\ e^x\\ -e^x\end{pmatrix}$$

is the corresponding solution for the system $[A]$.

Remark 2.3.14. The algebrization given in proposition 2.3.9 is an example of how the introduction of the new derivative $\widehat{\partial}_z$ simplifies the proofs and computations. Such proposition is naturally extended to $\partial_x^2 y + a\partial_x y + by = 0$, using φ to obtain $\widehat{\partial}_z^2\widehat{y} + \widehat{a}\widehat{\partial}_z\widehat{y} + \widehat{b}\widehat{y} = 0$, which is equivalent to

$$\alpha\partial_z^2\widehat{y} + \left(\frac{\partial_x\alpha}{2} + \sqrt{\alpha}\widehat{a}\right)\partial_z\widehat{y} + \widehat{b}\widehat{y} = 0, \tag{2.8}$$

where $y(x) = \widehat{y}(z(x))$, $\widehat{a}(z(x)) = a(x)$ and $\widehat{b}(z(x)) = b(x)$.

In general, for $y(x) = \widehat{y}(z(x))$, the equation $F(\partial_x^n y, \dots, y, x) = 0$ with coefficients given by $a_{i_k}(x)$ is transformed in the equation $\widehat{F}(\widehat{\partial}_z^n\widehat{y}, \dots, \widehat{y}, z) = 0$ with coefficients given by $\widehat{a}_{i_k}(z)$, where $a_{i_k}(x) = \widehat{a}_{i_k}(z(x))$. In particular, for $\sqrt{\alpha}, \widehat{a}_{i_k} \in \mathbb{C}(z)$, the equation $\widehat{F}(\widehat{\partial}_z^n\widehat{y}, \dots, \widehat{y}, z) = 0$ is the Hamiltonian algebrization of $F(\partial_x^n y, \dots, y, x) = 0$. Now, if each derivation ∂_x has order even, then α and $\widehat{a}_{i_k}$ can be rational functions to algebrize the equation $F(\partial_x^n y, \dots, y, x) = 0$, where $a_{i_k} \in \mathbb{C}(z(x), \partial_x z(x))$. for example, that happens for linear differential equations given by

$$\partial_x^{2n} y + a_{n-1}(x)\partial_x^{2n-2} y + \ldots + a_2(x)\partial_x^4 y + a_1(x)\partial_x^2 y + a_0(x)y = 0.$$

Finally, the algebrization algorithm given in proposition 2.3.11 can be naturally extended to any differential equation

$$F(\partial_x^n y, \partial_x^{n-1} y, \dots, \partial_x y, y, e^{\mu t}) = 0,$$

that by means of the change of variable $z = e^{\mu x}$ is transformed into

$$\widehat{F}(\widehat{\partial}_z^n\widehat{y}, \widehat{\partial}_z^{n-1}\widehat{y}, \dots, \widehat{\partial}_z\widehat{y}, y, z) = 0.$$

Particularly, we consider the algebrization of Riccati equations, higher order linear differential equations and systems.

Examples. The following corresponds to some examples of algebrizations for differential equations given in [92, p. 258, 266].

1. The equation $\mathcal{L} := \partial_x^2 y + (-2e^x - 1)\partial_x y + e^{2x}y = 0$ with the Hamiltonian change of variable $z = e^x$, $\sqrt{\alpha} = z$, $\widehat{a} = -2z - 1$ and $\widehat{b} = z^2$ is transformed in the equation $\widehat{\mathcal{L}} := \partial_z^2\widehat{y} - 2\partial_z\widehat{y} + \widehat{y} = 0$ which can be easily solved. A basis of solutions for $\mathcal{L}$ and $\widehat{\mathcal{L}}$ are given by $\langle e^z, ze^z\rangle$ and $\langle e^{e^x}, e^x e^{e^x}\rangle$ respectively. Furthermore $K = \mathbb{C}(e^x)$, $\widehat{K} = \mathbb{C}(z)$, L and $\widehat{L}$ are the Picard-Vessiot extensions of $\mathcal{L}$ and $\widehat{\mathcal{L}}$ respectively. Thus, $\mathrm{DGal}(L/K) = \mathrm{DGal}(\widehat{L}/\widehat{K})$.

2. The differential equation

$$\mathcal{L} := \partial_x^2 y + \frac{-24e^x - 25}{4e^x + 5}\partial_x y + \frac{20e^x}{4e^x + 5}y = 0$$

with the Hamiltonian change of variable $z = e^x$, $\sqrt{\alpha} = z$,

$$\widehat{a} = \frac{-24z - 25}{4z + 5} \text{ and } \widehat{b} = \frac{20z}{4z + 5}$$

is transformed in the equation

$$\widehat{\mathcal{L}} := \partial_z^2\widehat{y} + \frac{-20(z+1)}{x(4z+5)}\partial_z\widehat{y} + \frac{20}{z(4z+5)}\widehat{y} = 0,$$

which can be solved with Kovacic algorithm. A basis of solutions for $\widehat{\mathcal{L}}$ is $\langle z + 1, z^5\rangle$, so that a basis for $\mathcal{L}$ is $\langle e^x + 1, e^{5x}\rangle$. Furthermore $K = \mathbb{C}(e^x)$, $\widehat{K} = \mathbb{C}(z)$, L and $\widehat{L}$ are the Picard-Vessiot extensions of $\mathcal{L}$ and $\widehat{\mathcal{L}}$ respectively. Thus, $\mathrm{DGal}(L/K) = \mathrm{DGal}(\widehat{L}/\widehat{K}) = e$.

Remark 2.3.15 (Algebrization of the Riccati equation). The Riccati equation

$$\partial_x v = a(x) + b(x)v + c(x)v^2 \tag{2.9}$$

through the Hamiltonian change of variable $z = z(x)$, becomes in the Riccati equation

$$\partial_z\widehat{v} = \frac{1}{\sqrt{\alpha}}(\widehat{a}(z) + \widehat{b}(z)\widehat{v} + \widehat{c}(z)\widehat{v}^2), \tag{2.10}$$

where $v(x) = \widehat{v}(z(x))$, $\widehat{a}(z(x)) = a(x)$, $\widehat{b}(z(x)) = b(x)$, $\widehat{c}(z(x)) = c(x)$ and $\sqrt{\alpha(z(x))} = \partial_x z(x)$. Furthermore, if $\sqrt{\alpha}$, $\widehat{a}$, $\widehat{b}$, $\widehat{c} \in \mathbb{C}(x)$, equation (2.10) is the algebrization of equation (2.9).

Example. Consider the Riccati differential equation

$$\mathcal{L} := \partial_x v = \left(\tanh x - \frac{1}{\tanh x}\right)v + \left(3\tanh x - 3\tanh^3 x\right)v^2,$$

which through the Hamiltonian change of variable $z = \tanh x$, for instance $\sqrt{\alpha} = 1 - z^2$, is transformed into the Riccati differential equation

$$\widehat{\mathcal{L}} := \partial_z v = -\frac{1}{z} v + 3zv^2.$$

One solution for the equation $\widehat{\mathcal{L}}$ is

$$-\frac{1}{z(3z-c)}, \text{ being } c \text{ a constant },$$

so that the corresponding solution for equation $\mathcal{L}$ is

$$-\frac{1}{\tanh x(3\tanh x - c)}.$$

The following result is the algebrized version of the relationship between the Eigenrings of systems and operators.

Proposition 2.3.16. *Consider the differential fields K, $\widehat{K}$ and consider the systems $[A]$ and $[\widehat{A}]$ given by*

$$\partial_x \mathbf{X} = -A\mathbf{X},\ \widehat{\partial}_z \widehat{\mathbf{X}} = -\widehat{A}\widehat{\mathbf{X}},\ \widehat{\partial}_z = \sqrt{\alpha}\partial_z,\ A = [a_{ij}],\ \widehat{A} = [\widehat{a}_{ij}],\ a_{ij} \in K,\ \widehat{a}_{ij} \in \widehat{K},$$

where $z = z(x)$, $\mathbf{X}(x) = \widehat{\mathbf{X}}(z(x))$, $\widehat{a}_{ij}(z(x)) = a_{ij}(x)$, then $\mathcal{E}(A) \simeq \mathcal{E}(\widehat{A})$. In particular, if we consider the linear differential equations

$$\mathcal{L} := \partial_x^n y + \sum_{k=0}^{n-1} a_k \partial_x^k y = 0 \quad \textit{and} \quad \widehat{\mathcal{L}} := \widehat{\partial}_z^n \widehat{y} + \sum_{k=0}^{n-1} \widehat{a}_k \widehat{\partial}_z^k \widehat{y} = 0,$$

where $z = z(x)$, $y(x) = \widehat{z}((x))$, $\widehat{a}_k(z(x)) = a_k(x)$, $a_k \in K$, $\widehat{a}_k \in \widehat{K}$, then $\mathcal{E}(\mathfrak{L}) \simeq \mathcal{E}(\widehat{\mathfrak{L}})$, where $\mathcal{L} := \mathfrak{L}y = 0$ and $\widehat{\mathcal{L}} := \widehat{\mathfrak{L}}\widehat{y} = 0$. Furthermore, assuming

$$P = \begin{pmatrix} p_{11} & \dots & p_{1n} \\ \vdots & & \\ p_{n1} & \dots & p_{nn} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & & & \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{pmatrix},$$

then

$$\mathcal{E}(\mathfrak{L}) = \left\{ \sum_{k=1}^{n} p_{1k} \partial_x^{k-1} : \partial_x P = PA - AP,\ p_{ik} \in K \right\},$$

if and only if

$$\mathcal{E}(\widehat{\mathfrak{L}}) = \left\{ \sum_{k=1}^{n} \widehat{p}_{1k} \widehat{\partial}_z^{k-1} : \widehat{\partial}_z \widehat{P} = \widehat{P}\widehat{A} - \widehat{A}\widehat{P},\ \widehat{p}_{ik} \in \widehat{K} \right\}.$$

Proof. By theorem 2.3.13 we have that $K \simeq \widehat{K}$, $\mathcal{E}(A) \simeq \mathcal{E}(\widehat{A})$ and $\mathcal{E}(\mathfrak{L}) \simeq \mathcal{E}(\widehat{\mathfrak{L}})$. Using the derivation $\widehat{\partial}_z$ and by induction on lemma 2.3.3 we complete the proof. □

Examples. We consider two different examples to illustrate the previous proposition.

- Consider the differential equation $\mathcal{L}_1 := \partial_x^2 y - (1 + \cos x - \cos^2 x)y = 0$. By means of the Hamiltonian change of variable $z = z(x) = \cos x$, with $\sqrt{\alpha} = -\sqrt{1 - z^2}$, $\mathcal{L}_1$ is transformed into the differential equation
$$\widehat{\mathcal{L}}_1 := \partial_z^2 \widehat{y} - \frac{z}{1 - z^2}\partial_z \widehat{y} - \frac{1 + z - z^2}{1 - z^2}\widehat{y} = 0.$$
Now, computing the eigenring of $\widehat{\mathfrak{L}}_1$ we have that $\mathcal{E}(\widehat{\mathfrak{L}}_1) = \mathrm{Vect}(1)$, therefore the eigenring of $\mathfrak{L}_1$ is given $\mathcal{E}(\mathfrak{L}_1) = \mathrm{Vect}(1)$.
- Now we consider the differential equation $\mathcal{L}_2 := \partial_x^2 y = \left(e^{2x} + \frac{9}{4}\right) y$. By means of the Hamiltonian change of variable $z = e^x$, with $\sqrt{\alpha} = x$, $\mathcal{L}_2$ is transformed into the differential equation
$$\widehat{\mathcal{L}}_2 := \partial_z^2 \widehat{y} + \frac{1}{z}\partial_z \widehat{y} - \left(1 + \frac{9}{4x^2}\right)\widehat{y} = 0.$$
Now, computing the eigenring of $\widehat{\mathfrak{L}}_2$ we have that
$$\mathcal{E}(\widehat{\mathfrak{L}}_2) = \mathrm{Vect}\left(1, -2\left(\frac{z^2 - 1}{z^2}\right)\partial_z - \frac{z^2 - 3}{z^3}\right) = \mathrm{Vect}\left(1, -2\left(\frac{z^2 - 1}{z^3}\right)\widehat{\partial}_z - \frac{z^2 - 3}{z^3}\right),$$
therefore the eigenring of $\mathfrak{L}_2$ is given by
$$\mathcal{E}(\mathfrak{L}_2) = \mathrm{Vect}\left(1, -2\left(\frac{e^{2x} - 1}{e^{3x}}\right)\partial_x - \frac{e^{2x} - 3}{e^{3x}}\right).$$
The same result is obtained via matrix formalism, where
$$A = \begin{pmatrix} 0 & 1 \\ e^{2x} + \frac{9}{4} & 0 \end{pmatrix}, \ \widehat{A} = \begin{pmatrix} 0 & 1 \\ z^2 + \frac{9}{4} & 0 \end{pmatrix}, \ \partial_x P = PA - AP, \ \widehat{\partial}_z \widehat{P} = \widehat{P}\widehat{A} - \widehat{A}\widehat{P},$$
with $P \in Mat(2, \mathbb{C}(e^x))$ and $\widehat{P} \in Mat(2, \mathbb{C}(z))$.

2.3.3 Applications in Supersymmetric Quantum Mechanics

In this subsection we apply the derivation $\widehat{\partial}_z$ to the Schrödinger equation $H\Psi = \lambda\Psi$, where $H = -\partial_x^2 + V(x)$, $V \in K$. Assume that $z = z(x)$ is a rational Hamiltonian change of variable for $H\Psi = \lambda\Psi$, then $K = \mathbb{C}(z(x), \partial_x z(x))$. Thus, the *algebrized Schrödinger equation* is written as

$$\widehat{H}\widehat{\Psi} = \lambda\widehat{\Psi}, \quad \widehat{H} = -\widehat{\partial}_z^2 + \widehat{V}(z), \quad \widehat{\partial}_z^2 = \alpha\partial_z^2 + \frac{1}{2}\partial_z\alpha\partial_z, \quad \widehat{K} = \mathbb{C}(z, \sqrt{\alpha}). \quad (2.11)$$

The *reduced algebrized Schrödinger equation*, obtained through equation (1.3), is given by

$$\widehat{\mathbf{H}}\Phi = \lambda\Phi, \quad \widehat{\mathbf{H}} = \alpha(z)\left(-\partial_z^2 + \widehat{\mathbf{V}}(z)\right), \quad \begin{array}{l} \widehat{\mathbf{V}}(z) = \mathcal{V} + \frac{\widehat{V}(z)}{\alpha}, \\ \mathcal{V} = \partial_z\mathcal{W} + \mathcal{W}^2, \\ \mathcal{W} = \frac{1}{4}\frac{\partial_z\alpha(z)}{\alpha(z)}. \end{array} \tag{2.12}$$

The eigenfunctions Ψ, $\widehat{\Psi}$ and Φ corresponding to the operators H, $\widehat{H}$ and $\widehat{\mathbf{H}}$ are related respectively as

$$\Phi(z(x)) = \sqrt[4]{\alpha}\widehat{\Psi}(z(x)) = \sqrt[4]{\alpha}\Psi(x).$$

In order to apply the Kovacic's algorithm we only consider the algebrized operator $\widehat{\mathbf{H}}$, whilst the eigenrings will be computed on $\widehat{H}$. Also it is possible to apply the version of Kovacic's algorithm given in reference [101] to the algebraized operator $\widehat{H}$. The following results are obtained by applying Kovacic's algorithm to the reduced algebrized Schrödinger equation (equation (2.12)) $\widehat{\mathbf{H}}\Phi = \lambda\Phi$.

Proposition 2.3.17. *Let $\widehat{\mathbf{L}}_\lambda$ be the Picard-Vessiot extension of the reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\Phi = \lambda\Phi$ with $\alpha, \widehat{V} \in \mathbb{C}[z]$. If $\deg\alpha < 2 + \deg\widehat{V}$, then* $\mathrm{DGal}(\widehat{\mathbf{L}}/\widehat{K})$ *is a not finite primitive for every $\lambda \in \Lambda$.*

Proof. Suppose that $\deg\alpha = n$ and $\deg\widehat{V} = m$. The reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\Phi = \lambda\Phi$ can be written in the form

$$\partial_z^2\Phi = r\Phi, \quad r = \frac{4\alpha\partial_z^2\alpha - 3(\partial_z\alpha)^2 + 16\alpha(\widehat{V} - \lambda)}{16\alpha^2}.$$

Due to $m > n - 2$ we have that $\circ(r_\infty) = n - m < 2$, which does not satisfy the condition (∞) of the case 3 of Kovacic's algorithm, therefore $\mathrm{DGal}(\widehat{\mathbf{L}}/\widehat{K})$ is a not finite primitive for every $\lambda \in \Lambda$.

□

Proposition 2.3.18. *Let $\widehat{\mathbf{L}}_\lambda$ be the Picard-Vessiot extension of the reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\Phi = \lambda\Phi$ with $\alpha \in \mathbb{C}[z]$, $\widehat{V} \in \mathbb{C}(z)$. If $\circ(\widehat{V})_\infty < 2 - \deg\alpha$, then* $\mathrm{DGal}(\widehat{\mathbf{L}}_\lambda/\widehat{K})$ *is a not finite primitive for every $\lambda \in \Lambda$.*

Proof. Suppose that $\widehat{V} = s/t$, being s and t co-primes polynomials in $\mathbb{C}(z)$. Assume that $\deg\alpha = n$, $\deg s = m$ and $\deg t = p$. The reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\Phi = \lambda\Phi$ can be written in the form

$$\partial_z^2\Phi = r\Phi, \quad r = \frac{4t\alpha\partial_z^2\alpha - 3t(\partial_z\alpha)^2 + 16\alpha(s - \lambda t)}{16t\alpha^2}.$$

Due to $m > n+p-2$ we have that $\circ(r_\infty) = p+n-m < 2$, which does not satisfy the condition (∞) of the case 3 of Kovacic's algorithm, therefore $\mathrm{DGal}(\widehat{\mathbf{L}}/\widehat{K})$ is a not finite primitive for every $\lambda \in \Lambda$.

□

Remark 2.3.19. In a natural way, we obtain the algebrized versions of Darboux transformation, i.e., the *algebrized Darboux transformation*, denoted by $\widehat{\mathrm{DT}}$. By $\widehat{\mathrm{DT}}_n$ we denote the n iteration of $\widehat{\mathrm{DT}}$, and by $\widehat{\mathrm{CI}}_n$ we denote the *algebrized Crum iteration*, where the *algebrized wronskian* is given by

$$\widehat{W}(\widehat{y}_1, \dots, \widehat{y}_n) = \begin{vmatrix} \widehat{y}_1 & \cdots & \widehat{y}_n \\ \vdots & & \vdots \\ \widehat{\partial}_z^{n-1}\widehat{y}_1 & \cdots & \widehat{\partial}_z^{n-1}\widehat{y}_n \end{vmatrix}.$$

In the same way, we define *algebrized shape invariant potentials*, *algebrized superpotential* $\widehat{W}$, *algebrized supersymmetric Hamiltonians* $\widehat{H}_\pm$, algebrized supersymmetric partner potentials $\widehat{V}_\pm$, *algebrized ground state* $\widehat{\Psi}_0^{(-)} = e^{-\int \frac{\widehat{W}}{\sqrt{\alpha}}dz}$, *algebrized wave functions* $\widehat{\Psi}_\lambda^{(-)}$, *algebrized raising and lowering operators* $\widehat{A}$ and $\widehat{A}^\dagger$. Thus, we can rewrite entirely the section 2.2 using the derivation $\widehat{\partial}_z$.

The following theorem show us the relationship between the algebrization and Darboux transformation.

Theorem 2.3.20. *Given the Schrödinger equation* $\mathcal{L}_\lambda := H_-\Psi^{(-)} = \lambda\Psi^{(-)}$, *the relationship between the algebrization* φ *and Darboux transformations* DT, $\widehat{\mathrm{DT}}$ *with respect to* $\mathcal{L}_\lambda$ *is given by* $\widehat{\mathrm{DT}}\varphi = \varphi\mathrm{DT}$, *that is* $\widehat{\mathrm{DT}}\varphi(\mathcal{L}) = \varphi\mathrm{DT}(\mathcal{L})$. *In other words, the Darboux transformations* DT *and* $\widehat{\mathrm{DT}}$ *are* intertwined *by the algebrization* φ.

Proof. Assume the equations $\mathcal{L}_\lambda := H_-\Psi^{(-)} = \lambda\Psi^{(-)}$, $\widehat{\mathcal{L}}_\lambda := \widehat{H}_-\widehat{\Psi}^{(-)} = \lambda\widehat{\Psi}^{(-)}$, $\widetilde{\mathcal{L}}_\lambda := H_+\Psi^{(+)} = \lambda\Psi^{(+)}$ and $\widetilde{\widehat{\mathcal{L}}} := \widehat{H}_+\widehat{\Psi}^{(+)} = \lambda\widehat{\Psi}^{(+)}$, where the Darboux transformations DT and $\widehat{\mathrm{DT}}$ are given by $\mathrm{DT}(\mathcal{L}) = \widetilde{\mathcal{L}}$, $\widehat{\mathrm{DT}}(\widehat{\mathcal{L}}) = \widetilde{\widehat{\mathcal{L}}}$,

$$\mathrm{DT}: \begin{array}{c} V_- \mapsto V_+ \\ \\ \Psi_\lambda^{(-)} \mapsto \Psi_\lambda^{(+)} \end{array}, \quad \widehat{\mathrm{DT}}: \begin{array}{c} \widehat{V}_- \mapsto \widehat{V}_+ \\ \\ \widehat{\Psi}_\lambda^{(-)} \mapsto \widehat{\Psi}_\lambda^{(+)}, \end{array}$$

and $\varphi(\mathcal{L}_\lambda) = \widehat{\mathcal{L}_\lambda}$, where the algebrization φ is given as in theorem 2.3.13. Then

the following diagram commutes

$$\begin{array}{ccc} \mathcal{L}_\lambda & \xrightarrow{\mathrm{DT}} & \widetilde{\mathcal{L}}_\lambda \\ \varphi\downarrow & & \downarrow\varphi \\ \widehat{\mathcal{L}}_\lambda & \xrightarrow[\widehat{\mathrm{DT}}]{} & \widehat{\widetilde{\mathcal{L}}}_\lambda \end{array} \qquad \Rightarrow \widehat{\mathrm{DT}}\varphi(\mathcal{L}) = \varphi \mathrm{DT}(\mathcal{L}) \Leftrightarrow \widetilde{\widehat{\mathcal{L}}} = \widehat{\widetilde{\mathcal{L}}}.$$

$\square$

To illustrate $\widehat{\mathrm{DT}}$ we present the following examples.

Examples. Consider the algebrized Schrödinger equation $\widehat{H}\widehat{\Psi}^{(-)} = \lambda\widehat{\Psi}^{(-)}$ with:

- $\sqrt{\alpha(z)} = \sqrt{z^2-1}$ and $\widehat{V}_-(z) = \frac{z}{z-1}$. Taking $\lambda_1 = 1$ and $\widehat{\Psi}_1^{(-)} = \sqrt{\frac{z+1}{z-1}}$, we have that $\widehat{\mathrm{DT}}(\widehat{V}_-) = \widehat{V}_+(z) = \frac{z}{z+1}$ and

$$\widehat{\mathrm{DT}}(\widehat{\Psi}_\lambda^{(-)}) = \widehat{\Psi}_\lambda^{(+)} = \sqrt{z^2-1}\partial_z\widehat{\Psi}_\lambda^{(-)} + \frac{1}{\sqrt{z^2-1}}\widehat{\Psi}_\lambda^{(-)},$$

  where $\widehat{\Psi}_\lambda^{(-)}$ is the general solution of $\widehat{H}_-\widehat{\Psi}^{(-)} = \lambda\widehat{\Psi}^{(-)}$ for $\lambda \neq 1$.

  The original potential corresponding to this example is given by $V_-(x) = \frac{\cosh x}{\cosh x - 1}$ and for $\lambda_1 = 1$ the particular solution $\Psi_1^{(-)}$ is given by $\frac{\sinh x}{\cosh x - 1}$. Applying DT we have that $\mathrm{DT}(V_-) = V_+(x) = \frac{\cosh x}{\cosh x+1}$ and $\mathrm{DT}(\Psi_\lambda^{(-)}) = \Psi_\lambda^{(+)} = \partial_x\Psi_\lambda^{(-)} + \frac{1}{\sinh x}\Psi_\lambda^{(-)}$.

- $\sqrt{\alpha} = -z$, $\widehat{V}_-(z) = z^2 - z$. Taking $\lambda_1 = 0$ and $\widehat{\Psi}_0^{(-)} = e^{-z}$ we have that $\widehat{\mathrm{DT}}(\widehat{V}_-) = V_+ = z^2 + z$ and $\widehat{\mathrm{DT}}(\widehat{\Psi}_\lambda^{(-)}) = \widehat{\Psi}_\lambda^{(+)} = -z\partial_z\widehat{\Psi}_\lambda^{(-)} - z\widehat{\Psi}_\lambda^{(-)}$, where $\Psi_\lambda^{(-)}$ is the general solution of $\widehat{H}_-\widehat{\Psi}^{(-)} = \lambda\widehat{\Psi}^{(-)}$ for $\lambda \neq 0$. This example corresponds to the Morse potential $V_-(x) = e^{-2x} - e^{-x}$, introduced in the list (1.40).

To illustrate $\widehat{\mathrm{CI}}_n$ we present the following example, which is related with the Chebyshev polynomials.

Example. Now, considering $\sqrt{\alpha} = -\sqrt{1-z^2}$, $V = 0$ with eigenvalues and eigenfunctions $\lambda_1 = 1$, $\lambda_2 = 4$, $\widehat{\Psi}_1 = z$, $\widehat{\Psi}_4 = 2z^2 - 1$, $\widehat{\Psi}_{n^2} = T_n(z)$, where $T_n(z)$ is the Chebyshev polynomial of first kind of degree n. The algebrized Wronskian for $n = 2$ is

$$\widehat{W}(z, 2z^2-1) = -\sqrt{1-z^2}(2z^2+1),$$

and by algebrized Crum iteration we obtain the potential

$$\widehat{\mathrm{CI}}_2(\widehat{V}) = \widehat{V}_2 = ((2z^2-1)\partial_z^2 + z\partial_z)\ln \widehat{W}(z, 2z^2-1)$$

and the algebrized wave functions

$$\widehat{\mathrm{CI}}_2(\widehat{\Psi}_\lambda) = \widehat{\Psi}_\lambda^{(2)} = \frac{\widehat{W}(z, 2z^2-1, T_n)}{\widehat{W}(z, 2z^2-1)}.$$

In a natural way we introduce the concept of algebrized shape invariant potentials $\widehat{V}_{n+1}(z, a_n) = \widehat{V}_n(z, a_{n+1}) + R(a_n)$, where the energy levels for $n > 0$ are given by $E_n = R(a_1) + \cdots R(a_n)$ and the algebrized eigenfunctions are given by $\widehat{\Psi}_n(a_1) = \widehat{A}^\dagger(z, a_1)\cdots \widehat{A}^\dagger(z, a_n)\Psi_0(z, a_n)$. To illustrate the algebrized shape invariant potentials and the operators $\widehat{A}$ and $\widehat{A}^\dagger$, we present the following example.

Example. Assume $\sqrt{\alpha} = 1 - z^2$ and the algebrized super potential $\widehat{W}(z) = z$. Following the method proposed in remark 2.2.16, step 1, we introduce $\mu \in \mathbb{C}$ to obtain $\widehat{W}(z;\mu) = \mu z$, and

$$\widehat{V}_\pm(z;\mu) = \widehat{W}^2(z;\mu) \pm \widehat{\partial}_z \widehat{W}(z;\mu) = \mu(\mu \mp 1)z^2 \pm \mu,$$

thus, $\widehat{V}_+(z;a_0) = a_0(a_0-1)z^2 + a_0$ and $\widehat{V}_-(z;a_1) = a_1(a_1+1)z^2 - a_1$. By step 2,

$$\widehat{\partial}_z(V_+(z;a_0) - V_-(z;a_1)) = 2z(1-z^2)(a_0(a_0-1) - a_1(a_1+1)).$$

By step 3, we obtain

$$a_1(a_1+1) = a_0(a_0-1), \quad a_1^2 - a_0^2 = -(a_1 + a_0),$$

and assuming $a_1 \neq \pm a_0$ we have $a_1 = f(a_0) = a_0 - 1$ and $R(a_1) = 2a_0 + 1 = (a_0+1)^2 - a_0^2 = a_1^2 - a_0^2$. This means that the potentials $\widehat{V}_\pm$ are algebrized shape invariant potentials where $E = E_n$ is easily obtained,

$$E_n = \sum_{k=1}^{n} R(a_k) = \sum_{k=1}^{n}\left(a_k^2 - a_{k-1}^2\right) = a_n^2 - a_0^2, \quad a_n = f^n(a_0) = a_0 + n.$$

Now, the algebrized ground state wave function of $\widehat{V}_-(z, a_0)$ is

$$\widehat{\Psi}_0 = e^{\int \frac{a_0 z}{1-z^2}dz} = \frac{1}{\left(\sqrt{1-z^2}\right)^{a_0}}.$$

Finally, we can obtain the rest of eigenfunctions using the algebrized raising operator:

$$\widehat{\Psi}_n(z, a_0) = \widehat{A}^\dagger(z, a_0)\widehat{A}^\dagger(z, a_1)\cdots \widehat{A}^\dagger(z, a_{n-1})\widehat{\Psi}_0(z, a_n).$$

This example corresponds to Pöschl-Teller potential introduced in the list (1.40).

Now to illustrate the power of Kovacic's algorithm with the derivation $\widehat{\partial}_z$, we study some Schrödinger equations for non-rational shape invariant potentials given in list (1.40). We work with specific values of these potentials, although we can apply our machinery (algebrization method and Kovacic's algorithm) using all the parameters of such potentials.

Morse potential: $V(x) = e^{-2x} - e^{-x}$.

The Schrödinger equation $H\Psi = \lambda\Psi$ is

$$\partial_x^2\Psi = \left(e^{-2x} - e^{-x} - \lambda\right)\Psi.$$

By the Hamiltonian change of variable $z = z(x) = e^{-x}$, we obtain

$$\alpha(z) = z^2, \quad \widehat{V}(z) = z^2 - z, \quad \widehat{\mathbf{V}}(z) = \frac{z^2 - z - \frac{1}{4}}{z^2}.$$

Thus, $\widehat{K} = \mathbb{C}(z)$ and $K = \mathbb{C}(e^x)$. In this way, the algebrized Schrödinger equation $\widehat{H}\widehat{\Psi} = \lambda\widehat{\Psi}$ is

$$z^2\partial_z^2\widehat{\Psi} + z\partial_z\widehat{\Psi} - (z^2 - z - \lambda)\widehat{\Psi} = 0$$

and the reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\widehat{\Phi} = \lambda\widehat{\Phi}$ is

$$\partial_z^2\Phi = r\Phi, \quad r = \frac{z^2 - z - \frac{1}{4} - \lambda}{z^2}.$$

This equation only could fall in case 1, in case 2 or in case 4 (of Kovacic's algorithm). We start analyzing the case 1: by conditions c_2 and ∞_3 we have that

$$\left[\sqrt{r}\right]_0 = 0, \quad \alpha_0^\pm = \frac{1 \pm 2\sqrt{-\lambda}}{2}, \quad \left[\sqrt{r}\right]_\infty = 1, \quad \text{and} \quad \alpha_\infty^\pm = \mp\frac{1}{2}.$$

By step 2 we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\begin{array}{lll}
\Lambda_{++}) & n = \alpha_\infty^+ - \alpha_0^+ = -1 - \sqrt{-\lambda}, & \lambda = -(n+1)^2, \\
\Lambda_{+-}) & n = \alpha_\infty^+ - \alpha_0^- = -1 + \sqrt{-\lambda}, & \lambda = -(n+1)^2, \\
\Lambda_{-+}) & n = \alpha_\infty^- - \alpha_0^+ = -\sqrt{-\lambda}, & \lambda = -n^2, \\
\Lambda_{--}) & n = \alpha_\infty^- - \alpha_0^- = \sqrt{-\lambda}, & \lambda = -n^2.
\end{array}$$

We can see that $\lambda \in \Lambda_- = \{-n^2 : n \in \mathbb{Z}_+\}$. Now, for $\lambda \in \Lambda$, the rational function

ω is given by:

$$
\begin{array}{llll}
\Lambda_{++}) & \omega = 1+\frac{3+2n}{2z}, & \lambda\in\Lambda_{++}, & r_n = \frac{4n^2+8n+3}{4z^2}+\frac{2n+3}{z}+1,\\
\Lambda_{+-}) & \omega = 1-\frac{1+2n}{2z}, & \lambda\in\Lambda_{+-}, & r_n = \frac{4n^2+8n+3}{4z^2}-\frac{2n+1}{z}+1,\\
\Lambda_{-+}) & \omega = -1+\frac{1+2n}{2z}, & \lambda\in\Lambda_{-+}, & r_n = \frac{4n^2-1}{4z^2}-\frac{2n+1}{z}+1,\\
\Lambda_{--}) & \omega = -1+\frac{1-2n}{2z}, & \lambda\in\Lambda_{--}, & r_n = \frac{4n^2-1}{4z^2}+\frac{2n-1}{z}+1,
\end{array}
$$

where r_n is the coefficient of the differential equation $\partial_z^2\Phi = r_n\Phi$.

By step 3, there exists a polynomial of degree n satisfying the relation (1.6),

$$
\begin{array}{llcl}
\Lambda_{++}) & \partial_z^2\widehat{P}_n + 2\left(1+\frac{3+2n}{2z}\right)\partial_z\widehat{P}_n + \frac{2(n+2)}{z}\widehat{P}_n & = & 0,\\
\Lambda_{+-}) & \partial_z^2\widehat{P}_n + 2\left(1-\frac{1+2n}{2z}\right)\partial_z\widehat{P}_n + \frac{2(-n)}{z}\widehat{P}_n & = & 0,\\
\Lambda_{-+}) & \partial_z^2\widehat{P}_n + 2\left(-1+\frac{1+2n}{2z}\right)\partial_z\widehat{P}_n + \frac{2(-n)}{z}\widehat{P}_n & = & 0,\\
\Lambda_{--}) & \partial_z^2\widehat{P}_n + 2\left(-1+\frac{1-2n}{2z}\right)\partial_z\widehat{P}_n + \frac{2n}{z}\widehat{P}_n & = & 0.
\end{array}
$$

These polynomials only exists for $n=\lambda=0$, with $\lambda\in\Lambda_{-+}\cup\Lambda_{--}$. So that the solutions of $H\Psi=0$, $\widehat{H}\widehat{\Psi}=0$ and $\widehat{\mathbf{H}}\Phi=0$ are given by

$$\Phi_0=\sqrt{z}e^{-z},\quad \widehat{\Psi}_0=e^{-z},\quad \Psi=e^{-e^{-x}}.$$

The wave function Ψ_0 satisfy the conditions (1.23), which means that is ground state (see [31]) and $0\in\operatorname{spec}_p(H)$. Furthermore, we have

$$\mathrm{DGal}(L_0/K)=\mathrm{DGal}(\widehat{L}_0/\widehat{K})=\mathrm{DGal}(\widehat{\mathbf{L}}_0/\mathbb{C}(z))=\mathbb{B},$$
$$\mathcal{E}(H)=\mathcal{E}(\widehat{H})=\mathcal{E}(\widehat{\mathbf{H}})=\mathrm{Vect}(1).$$

We follow with the case two. The conditions c_2 and ∞_3 are satisfied, in this way we have

$$E_c=\left\{2,2+4\sqrt{-\lambda},2-4\sqrt{-\lambda}\right\}\quad\text{and}\quad E_\infty=\{0\},$$

and by step two, we have that $2\pm\sqrt{-\lambda}=m\in\mathbb{Z}_+$, so that $\lambda=-\left(\frac{m+1}{2}\right)^2$ and the rational function θ has the following possibilities

$$\theta_+=\frac{2+m}{z},\quad \theta_-=-\frac{m}{z}.$$

By step three, there exist a monic polynomial of degree m satisfying the recurrence relation (1.7):

$$\begin{array}{lll} \theta_+) & \partial_z^3\widehat{P}_m+\frac{3m+6}{z}\partial_z^2\widehat{P}_m-\frac{4z^2-4z-2m^2-7m-6}{z^2}\partial_z\widehat{P}_m-\frac{4mz+8z-4m-6}{z^2}\widehat{P}_m & = & 0, \\ \theta_-) & \partial_z^3\widehat{P}_m-\frac{3m}{z}\partial_z^2\widehat{P}_m-\frac{4z^2-4z-2m^2-m}{z^2}\partial_z\widehat{P}_m+\frac{4mz-4m-2}{z^2}\widehat{P}_m & = & 0. \end{array}$$

We can see that for $m = 1$ the polynomial exists only for the case θ_-, being $\widehat{P}_1 = z - 1/2$. In general, these polynomials could exist only for the case θ_- with $m = 2n - 1$, $n \geq 1$, that is $\lambda \in \{-n^2 : n \geq 1\}$.

For instance, by case one and case two, we obtain $\Lambda = \{-n^2 : n \geq 0\} = \mathrm{spec}_p(H)$. Now, the rational function ϕ and the quadratic expression for ω are

$$\phi = -\frac{2n-1}{z} + \frac{\partial_z\widehat{P}_{2n-1}}{\widehat{P}_{2n-1}}, \quad \omega^2 + M\omega + N = 0, \quad \omega = \frac{-M \pm \sqrt{M^2-4N}}{2},$$

where the coefficients M and N are given by

$$M = \frac{2n-1}{z} - \frac{\partial_z\widehat{P}_{2n-1}}{\widehat{P}_{2n-1}}, \quad N = \frac{n^2-n+\frac{1}{4}}{z^2} - \frac{(2n-1)\frac{\partial_z\widehat{P}_{2n-1}}{\widehat{P}_{2n-1}} - 2}{z} + \frac{\partial_z^2\widehat{P}_{2n-1}}{\widehat{P}_{2n-1}} - 2.$$

Now, $\triangle = M^2 - 4N \neq 0$, which means that $\widehat{\mathbf{H}}\Phi = -n^2\Phi$ with $n \in \mathbb{Z}^+$ has two solutions given by Kovacic's algorithm:

$$\Phi_{1,n} = \frac{\sqrt{z}\widehat{P}_n e^{-z}}{z^n}, \quad \Phi_{2,n} = \frac{\sqrt{z}\widehat{P}_{n-1}e^{z}}{z^n}.$$

The solutions of $\widehat{H}\widehat{\Psi} = -n^2\widehat{\Psi}$ are given by

$$\widehat{\Psi}_{1,n} = \frac{\widehat{P}_n e^{-z}}{z^n}, \quad \widehat{\Psi}_{2,n} = \frac{\widehat{P}_{n-1}e^{z}}{z^n},$$

and therefore, the solutions of the Schrödinger equation $H\Psi = -n^2\Psi$ are

$$\Psi_{1,n} = P_n e^{-e^{-x}}e^{nx}, \quad \Psi_{2,n} = P_{n-1}e^{e^{-x}}e^{nx}, \quad P_n = \widehat{P}_n \circ z.$$

The wave functions $\Psi_{1,n} = \Psi_n$ satisfies the conditions of bound state, and for $n = 0$, this solution coincides with the ground state presented above. Therefore we have

$$\Phi_n = \Phi_0\widehat{f}_n\widehat{P}_n, \quad \widehat{\Psi}_n = \widehat{\Psi}_0\widehat{f}_n\widehat{P}_n, \quad \widehat{f}_n(z) = \frac{1}{z^n}.$$

Thus, the bound states wave functions are obtained as

$$\Psi_n = \Psi_0 f_n P_n, \quad f_n(x) = \widehat{f}_n(e^{-x}) = e^{nx}.$$

The Eigenrings and differential Galois groups for $n > 0$ satisfies

$$\mathrm{DGal}(L_n/K) = \mathrm{DGal}(\widehat{L}_n/\widehat{K}) = \mathrm{DGal}(\widehat{\mathbf{L}}_n/\mathbb{C}(z)) = \mathbb{G}_m,$$

$$\dim_{\mathbb{C}} \mathcal{E}(\widehat{\mathbf{H}} + n^2) = \dim_{\mathbb{C}} \mathcal{E}(\widehat{H} + n^2) = \dim_{\mathbb{C}} \mathcal{E}(H + n^2) = 2.$$

We remark that the Schrödinger equation with Morse potential, under suitable changes of variables [60], falls in a Bessel's differential equation. Thus we can obtain its integrability by means of corollary 1.1.32.

It is known that Eckart, Rosen-Morse, Scarf and Pöschl-Teller potentials, under suitable transformations, fall in an Hypergeometric equation which allows apply theorem 1.1.30. These potentials are inter-related by point canonical coordinate transformations (see [24, p. 314]), so that $\Lambda = \mathbb{C}$ due to Pöschl-Teller potential is obtained by means of Darboux transformations of $V = 0$ ([66, 86]). We consider some particular cases of Eckart, Scarf and Poschl-Teller potentials applying only the case 1 of Kovacic's algorithm. The case 1 allow us to obtain the enumerable set $\Lambda_n \subset \Lambda$, which include the classical results obtained by means of supersymmetric quantum mechanics. Cases 2 and 3 of Kovacic algorithm also can be applied, but are not considered here.

Eckart potential: $V(x) = 4\coth(x) + 5$, $x > 0$.

The Schrödinger equation $H\Psi = \lambda\Psi$ is

$$\partial_x^2 \Psi = (4\coth(x) + 5 - \lambda)\,\Psi.$$

By the Hamiltonian change of variable $z = z(x) = \coth(x)$, we obtain

$$\alpha(z) = (1 - z^2)^2, \quad \widehat{V}(z) = 4z + 5, \quad \widehat{\mathbf{V}}(z) = \frac{4}{(z+1)(z-1)^2}.$$

Thus, $\widehat{K} = \mathbb{C}(z)$ and $K = \mathbb{C}(\coth(x))$. In this way, the algebrized Schrödinger equation $\widehat{H}\widehat{\Psi} = \lambda\widehat{\Psi}$ is

$$(1 - z^2)^2 \partial_z^2 \widehat{\Psi} - 2z(1 - z^2)\partial_z \widehat{\Psi} - (4z + 5 - \lambda)\widehat{\Psi} = 0$$

and the reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\widehat{\Phi} = \lambda\widehat{\Phi}$ is

$$\partial_z^2 \Phi = r\Phi, \quad r = \frac{4z + 4 - \lambda}{(z-1)^2\,(z+1)^2} = \frac{2 - \frac{\lambda}{4}}{(z-1)^2} + \frac{\frac{\lambda}{4} - 1}{(z-1)} + \frac{-\frac{\lambda}{4}}{(z+1)^2} + \frac{1 - \frac{\lambda}{4}}{(z+1)}$$

We can see that this equation could fall in any case of Kovacic's algorithm. Considering $\lambda = 0$, the conditions $\{c_1, c_2, \infty_1\}$ of case 1 are satisfied, obtaining

$$[\sqrt{r}]_{-1} = [\sqrt{r}]_1 = [\sqrt{r}]_\infty = \alpha_\infty^+ = 0, \quad \alpha_{-1}^{\pm} = \alpha_\infty^- = 1, \quad \alpha_1^+ = 2, \quad \alpha_1^- = -1.$$

By step 2, the elements of D are 0 and 1. The rational function ω for $n=0$ and for $n=1$ must be
$$\omega=\frac{1}{z+1}+\frac{-1}{z-1}.$$
By step 3 we search the monic polynomial of degree n satisfying the relation (1.6). Starting with $n=0$ the only one possibility is $\widehat{P}_0(z)=1$, which effectively satisfy the relation (1.6), while $\widehat{P}_1(z)=z+a_0$ does not exists. In this way we have obtained one solution using Kovacic algorithm:
$$\Phi_0=\frac{z+1}{z-1},\quad \widehat{\Psi}_0=\sqrt{\frac{z+1}{(z-1)^3}},$$
this means that $0\in\Lambda_n$. We can obtain the second solution using the first solution:
$$\Phi_{0,2}=\frac{z^2+z-4-4\,\ln(z+1)\,z-4\,\ln(z+1)}{z-1},\quad \widehat{\Psi}_{0,2}=\frac{\Phi_{0,2}}{\sqrt{z^2-1}}.$$
Furthermore the differential Galois groups and Eigenrings for $\lambda=0$ are
$$\mathrm{DGal}(\widehat{\mathbf{L}}_0/\mathbb{C}(z))=\mathbb{G}_a,\quad \mathrm{DGal}(L_0/K)=\mathrm{DGal}(\widehat{L}_0/\widehat{K})=\mathbb{G}^{\{2\}},$$
$$\mathcal{E}(\widehat{\mathbf{H}})=\mathrm{Vect}\left(1,\frac{(z+1)^2}{(1-z)^2}\partial_z+\frac{2(z+1)}{(1-z)^3}\right),$$
$$\mathcal{E}(\widehat{H})=\mathrm{Vect}\left(1,\frac{(z+1)^2}{(1-z)^2}\partial_z-\frac{z^2+3z+2}{(1-z)^3}\right),$$
$$\mathcal{E}(H)=\mathrm{Vect}\left(1,\frac{(\coth(x)+1)^2}{(1-\coth(x))^2(1-\coth^2(x))}\partial_x-\frac{\coth^2(x)+3\coth(x)+2}{(1-\coth(x))^3}\right).$$
Now, for $\lambda\neq 0$, the conditions $\{c_2,\infty_1\}$ of case 1 are satisfied:
$$[\sqrt{r}]_{-1}=[\sqrt{r}]_1=[\sqrt{r}]_\infty=\alpha_\infty^+=0,\quad \alpha_\infty^-=1,$$
$$\alpha_{-1}^\pm=\tfrac{1\pm\sqrt{1-\lambda}}{2},\quad \alpha_1^\pm=\tfrac{1\pm\sqrt{9-\lambda}}{2}.$$
By step 2 we have the following possibilities for $n\in\mathbb{Z}_+$ and for $\lambda\in\Lambda$:

$\Lambda_{++-})\quad n=\alpha_\infty^+-\alpha_{-1}^+-\alpha_1^-=-1-\tfrac{\sqrt{1-\lambda}-\sqrt{9-\lambda}}{2},\quad \lambda=4-\tfrac{4}{(n+1)^2}-n^2-2n,$

$\Lambda_{+--})\quad n=\alpha_\infty^+-\alpha_{-1}^--\alpha_1^-=-1+\tfrac{\sqrt{1-\lambda}+\sqrt{9-\lambda}}{2},\quad \lambda=4-\tfrac{4}{(n+1)^2}-n^2-2n,$

$\Lambda_{-+-})\quad n=\alpha_\infty^--\alpha_{-1}^+-\alpha_1^-=\tfrac{\sqrt{1-\lambda}+\sqrt{9-\lambda}}{2},\quad \lambda=5-\tfrac{4}{n^2}-n^2,$

$\Lambda_{---})\quad n=\alpha_\infty^--\alpha_{-1}^--\alpha_1^-=\tfrac{\sqrt{1-\lambda}+\sqrt{9-\lambda}}{2},\quad \lambda=5-\tfrac{4}{n^2}-n^2.$

Therefore, we have that

$$\Lambda_n \subseteq \left\{4 - \frac{4}{(n+1)^2} - n^2 - 2n : n \in \mathbb{Z}_+\right\} \cup \left\{5 - \frac{4}{n^2} - n^2 : n \in \mathbb{Z}_+\right\}.$$

Now, for $\lambda \in \Lambda$, the rational function ω is given by:

$$\Lambda_{++-}) \qquad \omega = \frac{z(n-1)-n^2-2n-1}{(n+1)(z+1)(z-1)}, \quad r_n = \frac{-2z^2(n-1)+4z(n+1)^2+(n+1)(n^3+3n^2+2n+2)}{(n+1)^2(z+1)^2(z-1)^2},$$

$$\Lambda_{+--}) \qquad \omega = \frac{nz(n+1)+2}{(n+1)(z+1)(1-z)}, \quad r_n = \frac{nz^2(n+1)^3+4z(n+1)^2+n^3+2n^2+n+4}{(n+1)^2(z+1)^2(z-1)^2},$$

$$\Lambda_{-+-}) \qquad \omega = \frac{z(n-2)-n^2}{n(z+1)(z-1)}, \qquad r_n = \frac{-2z^2(n-2)+4n^2z+n(n^3-n+2)}{n^2(z+1)^2(z-1)^2},$$

$$\Lambda_{---}) \qquad \omega = \frac{nz(n-1)+2}{n(z+1)(1-z)}, \qquad r_n = \frac{n^3z^2(n-1)+4n^2z+n^3-n^2+4}{n^2(z+1)^2(z-1)^2},$$

where r_n is the coefficient of the differential equation $\partial_z^2\Phi = r_n\Phi$.

By step 3, there exists a monic polynomial of degree n satisfying the relation (1.6),

$$\begin{array}{llcl}
\Lambda_{++-}) & \partial_z^2\widehat{P}_n + 2\left(\frac{z(n-1)-n^2-2n-1}{(n+1)(z+1)(z-1)}\right)\partial_z\widehat{P}_n + \frac{2(1-n)}{((n+1)^2(z+1)(z-1)}\widehat{P}_n & = & 0,\\
\Lambda_{+--}) & \partial_z^2\widehat{P}_n + 2\left(\frac{nz(n+1)+2}{(n+1)(z+1)(1-z)}\right)\partial_z\widehat{P}_n + \frac{n(n+1)}{(z+1)(z-1)}\widehat{P}_n & = & 0,\\
\Lambda_{-+-}) & \partial_z^2\widehat{P}_n + 2\left(\frac{z(n-2)-n^2}{n(z+1)(z-1)}\right)\partial_z\widehat{P}_n + \frac{2(2-n)}{n^2(z+1)(z-1)}\widehat{P}_n & = & 0,\\
\Lambda_{---}) & \partial_z^2\widehat{P}_n + 2\left(\frac{nz(n-1)+2}{n(z+1)(1-z)}\right)\partial_z\widehat{P}_n + \frac{n(n-1)}{(z+1)(z-1)}\widehat{P}_n & = & 0.
\end{array}$$

The only one case in which there exist the polynomial $\widehat{P}_n$ of degree n is for Λ_{+--}). The solutions of the equation $\widehat{\mathbf{H}}\Phi = \lambda\Phi$, with $\lambda \neq 0$, are:

$$\Lambda_{++-}) \qquad \Phi_n = \widehat{P}_n\widehat{f}_n\Phi_0, \quad \Phi_0 = \frac{1}{z-1} \quad \widehat{f}_n = (z-1)^{\frac{n(1-n)}{2(n+1)}}(z+1)^{\frac{n(n+3)}{2(n+1)}},$$

$$\Lambda_{+--}) \qquad \Phi_n = \widehat{P}_n\widehat{f}_n\Phi_0, \quad \Phi_0 = \frac{z+1}{z-1} \quad f_n = (z-1)^{\frac{n(1-n)}{2(n+1)}}(z+1)^{\frac{-n(n+3)}{2(n+1)}},$$

$$\Lambda_{-+-}) \qquad \Phi_n = \widehat{P}_n\widehat{f}_n\Phi_1, \quad \Phi_1 = \frac{1}{z-1} \quad \widehat{f}_n = (z+1)^{\frac{n^2+n-2}{2n}}(z-1)^{\frac{-n^2+3n-2}{2n}},$$

$$\Lambda_{---}) \qquad \Phi_n = \widehat{P}_n\widehat{f}_n\Phi_1, \quad \Phi_1 = \frac{z+1}{z-1} \quad \widehat{f}_n = (z+1)^{\frac{-n^2-n+2}{2n}}(z-1)^{\frac{-n^2+3n-2}{2n}}.$$

In any case $\widehat{\Psi}_n = \frac{\Phi_n}{1-z^2}$, but the case Λ_{+--}) includes the classical results obtained by means of supersymmetric quantum mechanics. Thus, replacing z by $\coth(x)$ we

obtain the eigenstates Ψ_n. The Eigenrings and differential Galois groups for $n > 0$ and $\lambda \in \Lambda_n$ satisfies

$$\mathrm{DGal}(L_\lambda/K) \subseteq \mathbb{G}^{\{2m\}}, \quad \mathrm{DGal}(\widehat{L}_\lambda/\widehat{K}) \subseteq \mathbb{G}^{\{2m\}},$$

$$\mathrm{DGal}(\widehat{\mathbf{L}}_\lambda/\mathbb{C}(z)) = \mathbb{G}_m,$$

$$\dim_{\mathbb{C}(z)} \mathcal{E}(\widehat{\mathbf{H}} + \lambda) = 2, \quad \mathcal{E}(\widehat{H} + \lambda) = \mathcal{E}(H + \lambda) = \mathrm{Vect}(1).$$

Scarf potential: $V(x) = \frac{\sinh^2 x - 3\sinh x}{\cosh^2 x}$.

The Schrödinger equation $H\Psi = E\Psi$ is

$$\partial_x^2 \Psi = \left(\frac{\sinh^2 x - 3\sinh x}{\cosh^2 x} - E \right) \Psi.$$

By the Hamiltonian change of variable $z = z(x) = \sinh(x)$, we obtain

$$\alpha(z) = 1 + z^2, \quad \widehat{V}(z) = \frac{z^2 - 3z}{1 + z^2}.$$

Thus, $\widehat{K} = \mathbb{C}(z, \sqrt{1+z^2})$ and $K = \mathbb{C}(\sinh(x), \cosh(x))$. In this way, the reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\Phi = \lambda\Phi$ is

$$\partial_z^2 = \left(\frac{\lambda z^2 - 12z + \lambda - 1}{4(z^2+1)^2} \right) \Phi, \quad \lambda = 3 - 4E.$$

Applying Kovacic's algorithm for this equation with $\lambda = 0$, we see that does not falls in case 1. We consider only $\lambda \neq 0$. By conditions $\{c_2, \infty_2\}$ of case 1 we have that

$$[\sqrt{r}]_{-\mathrm{i}} = [\sqrt{r}]_{\mathrm{i}} = [\sqrt{r}]_\infty = 0, \quad \alpha_\infty^\pm = \tfrac{1 \pm \sqrt{1+\lambda}}{2},$$

$$\alpha_{-\mathrm{i}}^+ = \tfrac{5}{4} - \tfrac{\mathrm{i}}{2}, \quad \alpha_{-\mathrm{i}}^- = -\tfrac{1}{4} + \tfrac{\mathrm{i}}{2} \quad \alpha_{\mathrm{i}}^+ = \tfrac{5}{4} + \tfrac{\mathrm{i}}{2}, \quad \alpha_{\mathrm{i}}^- = -\tfrac{1}{4} - \tfrac{\mathrm{i}}{2}.$$

By step 2 we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\Lambda_{+++}) \quad n = \alpha_\infty^+ - \alpha_{-\mathrm{i}}^+ - \alpha_{\mathrm{i}}^+ = \tfrac{\sqrt{\lambda+1}-4}{2}, \quad \lambda = 4n^2 + 16n + 15,$$

$$\Lambda_{+--}) \quad n = \alpha_\infty^+ - \alpha_{-\mathrm{i}}^- - \alpha_{\mathrm{i}}^- = \tfrac{\sqrt{\lambda+1}+2}{2}, \quad \lambda = 4n^2 - 8n + 3,$$

obtaining in this way

$$\Lambda_n \subseteq \left\{4n^2 + 16n + 15 : n \in \mathbb{Z}_+\right\} \cup \left\{4n^2 - 8n + 3 : n \in \mathbb{Z}_+\right\}.$$

Now, the rational function ω is given by:

$$\Lambda_{+++})\qquad \omega=\frac{5z-2}{2(z^2+1)},\qquad \Lambda_{+--})\quad \frac{2-z}{2(z^2+1)}.$$

By step 3, there exists $\widehat{P}_0=1$ and a polynomial of degree $n\geq 1$ should satisfy either of the relation (1.6),

$$\begin{array}{lrcl}\Lambda_{+++}) & \partial_z^2\widehat{P}_n+\frac{5z-2}{z^2+1}\partial_z\widehat{P}_n-\frac{nz^2(n+4)+3z+n^2+4n-3}{(z^2+1)^2}\widehat{P}_n & = & 0,\\ \Lambda_{+--}) & \partial_z^2\widehat{P}_n+\frac{2-z}{z^2+1}\partial_z\widehat{P}_n-\frac{nz^2(n-2)+3z+n^2-2n-3}{(z^2+1)^2}\widehat{P}_n & = & 0.\end{array}$$

In both cases there exists the polynomial $\widehat{P}_n$ of degree $n\geq 1$. Basis of solutions $\{\Phi_{1,n},\Phi_{2,n}\}$ of the reduced algebrized Schrödinger equation are:

$$\begin{array}{llll}\Lambda_{+++}) & \Phi_{1,n}=\widehat{P}_n\widehat{f}_n\Phi_{1,0}, & \Phi_{1,0}=(1+z^2)^{\frac{5}{4}}e^{-\arctan z}, & \widehat{f}_n=1,\\ & \Phi_{2,n}=\widehat{Q}_n\widehat{g}_n\Phi_{2,0}, & \Phi_{2,0}=\frac{22+21x+12x^2+6x^3}{\sqrt[4]{1+z^2}}e^{-\arctan z}, & \widehat{g}_n=1.\\ \Lambda_{+--}) & \Phi_{1,n}=\widehat{P}_n\widehat{f}_n\Phi_{1,0}, & \Phi_{1,0}=\frac{1}{\sqrt[4]{1+z^2}}e^{\arctan z}, & \widehat{f}_n=1,\\ & \Phi_{2,n}=\widehat{Q}_n\widehat{g}_n\Phi_{2,0}, & \Phi_{2,0}=\frac{1}{\sqrt[4]{1+z^2}}e^{\arctan z}\int\frac{1}{\sqrt{1+z^2}}e^{-2\arctan z}dz, & \widehat{g}_n=1.\end{array}$$

In both cases $\widehat{\Psi}=\frac{\Phi}{\sqrt[4]{1+z^2}}$, but the classical case (see references [24, 31]) is Λ_{+--}), so that replacing z by $\sinh x$ and λ by $3-4E$ we obtain the eigenstates Ψ_n.

The Eigenrings and differential Galois groups are

$$\mathcal{E}(H-\lambda)=\mathcal{E}(\widehat{H}-\lambda)=\mathcal{E}(\widehat{\mathbf{H}}-\lambda)=\mathrm{Vect}(1),$$
$$\mathrm{DGal}(L_\lambda/K)=\mathrm{DGal}(\widehat{L}_\lambda/\widehat{K})=\mathrm{DGal}(\widehat{\mathbf{L}}_\lambda/\mathbb{C}(x))=\mathbb{B}.$$

Pöschl-Teller potential: $V(\mathrm{r})=\frac{\cosh^4(x)-\cosh^2(x)+2}{\sinh^2(x)\cosh^2(x)}$, $x>0$. The reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\Phi=E\Phi$ is

$$\partial_z^2\Phi=\left(\frac{\lambda z^4-(\lambda+3)z^2+8}{4z^2(z^2-1)^2}\right)\Phi,\quad \lambda=3-4E.$$

Considering $\lambda=0$ and starting with the conditions $\{c_2,\infty_1\}$ of case 1, we obtain

$$[\sqrt{r}]_0=[\sqrt{r}]_{-1}=[\sqrt{r}]_1=[\sqrt{r}]_\infty=\alpha_\infty^+=0,\quad \alpha_\infty^-=1,$$
$$\alpha_{-1}^+=\alpha_1^+=\tfrac{5}{4},\quad \alpha_{-1}^-=\alpha_1^-=-\tfrac{1}{4},\quad \alpha_0^+=2,\quad \alpha_0^-=-1.$$

By step 2, the elements of D are 0 and 1. The rational function ω has the following possibilities for $n = 0$ and for $n = 1$:

$$\Lambda_{++--}) \quad n = 0, \quad \omega = \tfrac{5/4}{z+1} + \tfrac{-1/4}{z-1} + \tfrac{-1}{z},$$

$$\Lambda_{+-+-}) \quad n = 0, \quad \omega = \tfrac{-1/4}{z+1} + \tfrac{5/4}{z-1} + \tfrac{-1}{z},$$

$$\Lambda_{-+--}) \quad n = 1, \quad \omega = \tfrac{5/4}{z+1} + \tfrac{-1/4}{z-1} + \tfrac{-1}{z},$$

$$\Lambda_{--+-}) \quad n = 1, \quad \omega = \tfrac{-1/4}{z+1} + \tfrac{5/4}{z-1} + \tfrac{-1}{z}.$$

By step 3 we search the monic polynomial of degree n satisfying the relation (1.6). Starting with $n = 0$ the only one possibility for $\Lambda_{++--})$ and $\Lambda_{+-+-})$ is $\widehat{P}_0(z) = 1$, which does not satisfy the relation (1.6) in both cases, while $\widehat{P}_1(z) = z + a_0$ effectively does exists, in where $a_0 = -\frac{2}{3}$ for $\Lambda_{-+--})$ and $a_0 = \frac{2}{3}$ for $\Lambda_{--+-})$. In this way we have obtained two solutions $(\Phi_{1,0}, \Phi_{2,0})$ using Kovacic's algorithm:

$$\Phi_{1,0} = \left(1 - \tfrac{2}{3z}\right) \sqrt[4]{\tfrac{(z+1)^5}{z-1}}, \quad \widehat{\Psi}_{1,0} = \left(1 - \tfrac{2}{3z}\right) \tfrac{z+1}{\sqrt{z-1}},$$

$$\Phi_{2,0} = \left(1 + \tfrac{2}{3z}\right) \sqrt[4]{\tfrac{(z-1)^5}{z+1}}, \quad \widehat{\Psi}_{2,0} = \left(1 + \tfrac{2}{3z}\right) \tfrac{z-1}{\sqrt{z+1}},$$

this means that $0 \in \Lambda_n$. Furthermore,

$$\mathrm{DGal}(\widehat{\mathbf{L}}_0/\mathbb{C}(x)) = \mathbb{G}^{[4]}, \quad \mathrm{DGal}(\widehat{L}_0/\widehat{K}) = \mathrm{DGal}(L_0/K) = e,$$

$$\dim_{\mathbb{C}} \mathcal{E}(\widehat{\mathbf{H}}) = 2, \quad \dim_{\mathbb{C}} \mathcal{E}(\widehat{H}) = \dim_{\mathbb{C}} \mathcal{E}(H) = 4.$$

Now, for $\lambda \neq 0$ we see that conditions $\{c_2, \infty_1\}$ of case 1 leads us to

$$[\sqrt{r}]_0 = [\sqrt{r}]_{-1} = [\sqrt{r}]_1 = [\sqrt{r}]_\infty = 0, \qquad \alpha_\infty^\pm = \tfrac{1 \pm \sqrt{1+\lambda}}{2},$$

$$\alpha_{-1}^+ = \alpha_1^+ = \tfrac{5}{4}, \quad \alpha_{-1}^- = \alpha_1^- = -\tfrac{1}{4}, \quad \alpha_0^+ = 2, \quad \alpha_0^- = -1.$$

By step 2 we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\Lambda_{++++}) \quad n = \alpha_\infty^+ - \alpha_{-1}^+ - \alpha_1^+ - \alpha_0^+ = \tfrac{\sqrt{\lambda+1}-8}{2}, \quad \lambda = 4n^2 + 32n + 63,$$

$$\Lambda_{+++-}) \quad n = \alpha_\infty^+ - \alpha_{-1}^+ - \alpha_1^+ - \alpha_0^- = \tfrac{\sqrt{\lambda+1}-2}{2}, \quad \lambda = 4n^2 + 8n + 3,$$

$$\Lambda_{++-+}) \quad n = \alpha_\infty^+ - \alpha_{-1}^+ - \alpha_1^- - \alpha_0^+ = \tfrac{\sqrt{\lambda+1}-5}{2}, \quad \lambda = 4n^2 + 20n + 24,$$

$$\Lambda_{++--}) \quad n = \alpha_\infty^+ - \alpha_{-1}^+ - \alpha_1^- - \alpha_0^- = \tfrac{\sqrt{\lambda+1}+1}{2}, \quad \lambda = 4n^2 - 4n,$$

$$\Lambda_{+-++}) \quad n = \alpha_\infty^+ - \alpha_{-1}^- - \alpha_1^+ - \alpha_0^+ = \tfrac{\sqrt{\lambda+1}-5}{2}, \quad \lambda = 4n^2 + 20n + 24,$$

$$\Lambda_{+-+-}) \quad n = \alpha_\infty^+ - \alpha_{-1}^- - \alpha_1^+ - \alpha_0^- = \tfrac{\sqrt{\lambda+1}+1}{2}, \quad \lambda = 4n^2 - 4n,$$

$$\Lambda_{+--+}) \quad n = \alpha_\infty^+ - \alpha_{-1}^- - \alpha_1^- - \alpha_0^+ = \tfrac{\sqrt{\lambda+1}-2}{2}, \quad \lambda = 4n^2 + 8n + 3,$$

$$\Lambda_{+---}) \quad n = \alpha_\infty^+ - \alpha_{-1}^- - \alpha_1^- - \alpha_0^- = \tfrac{\sqrt{\lambda+1}+4}{2}, \quad \lambda = 4n^2 - 16n + 15,$$

obtaining $\Lambda_n \subseteq \Lambda_a \cup \Lambda_b \cup \Lambda_c \cup \Lambda_d \cup \Lambda_e$, where

$$\Lambda_a = \left\{4n^2 + 32n + 63 : n \in \mathbb{Z}_+\right\}, \quad \Lambda_b = \left\{4n^2 + 8n + 3 : n \in \mathbb{Z}_+\right\}$$
$$\Lambda_c = \left\{4n^2 + 20n + 24 : n \in \mathbb{Z}_+\right\}, \quad \Lambda_d = \left\{4n^2 - 4n : n \in \mathbb{Z}_+\right\},$$
$$\Lambda_e = \left\{4n^2 - 16n + 15 : n \in \mathbb{Z}_+\right\}.$$

Now, the rational function ω is given by:

$$\Lambda_{++++}) \quad \omega = \tfrac{5/4}{z+1} + \tfrac{5/4}{z-1} + \tfrac{2}{z}, \qquad \Lambda_{+++-}) \quad \omega = \tfrac{5/4}{z+1} + \tfrac{5/4}{z-1} + \tfrac{-1}{z},$$

$$\Lambda_{++-+}) \quad \omega = \tfrac{5/4}{z+1} + \tfrac{-1/4}{z-1} + \tfrac{2}{z}, \qquad \Lambda_{++--}) \quad \omega = \tfrac{5/4}{z+1} + \tfrac{-1/4}{z-1} + \tfrac{-1}{z},$$

$$\Lambda_{+-++}) \quad \omega = \tfrac{-1/4}{z+1} + \tfrac{5/4}{z-1} + \tfrac{2}{z}, \qquad \Lambda_{+-+-}) \quad \omega = \tfrac{-1/4}{z+1} + \tfrac{5/4}{z-1} + \tfrac{-1}{z},$$

$$\Lambda_{+--+}) \quad \omega = \tfrac{-1/4}{z+1} + \tfrac{-1/4}{z-1} + \tfrac{2}{z}, \qquad \Lambda_{+---}) \quad \omega = \tfrac{-1/4}{z+1} + \tfrac{-1/4}{z-1} + \tfrac{-1}{z}.$$

By step 3, there exists a monic polynomial of degree n satisfying the relation

(1.6),

$$\Lambda_{++++})\quad \partial_z^2\widehat{P}_n+\tfrac{(3z+2)(3z-2)}{z(z+1)(z-1)}\partial_z\widehat{P}_n+\tfrac{n(n+8)}{(z+1)(1-z)}P_n=0,$$

$$\Lambda_{+++-})\quad \partial_z^2\widehat{P}_n+\tfrac{3z^2+2}{z(z+1)(z-1)}\partial_z\widehat{P}_n+\tfrac{n(n+2)}{(z+1)(1-z)}\widehat{P}_n=0,$$

$$\Lambda_{++-+})\quad \partial_z^2\widehat{P}_n+\tfrac{6z^2-3x-4}{z(z+1)(z-1)}\partial_z\widehat{P}_n+\tfrac{nz(n+5)+6}{z(z+1)(1-z)}\widehat{P}_n=0,$$

$$\Lambda_{++--})\quad \partial_z^2\widehat{P}_n+\tfrac{3z-2}{z(z+1)(1-z)}\partial_z\widehat{P}_n+\tfrac{nz(n-1)-12}{4z(z+1)(1-z)}\widehat{P}_n=0,$$

$$\Lambda_{+-++})\quad \partial_z^2\widehat{P}_n+\tfrac{6z^2+3z-4}{z(z+1)(z-1)}\partial_z\widehat{P}_n+\tfrac{z(n^2+5)-6}{z(z+1)(1-z)}\widehat{P}_n=0,$$

$$\Lambda_{+-+-})\quad \partial_z^2\widehat{P}_n+\tfrac{3z+2}{z(z+1)(z-1)}\partial_z\widehat{P}_n+\tfrac{nz(n-1)+3}{z(z+1)(1-z)}\widehat{P}_n=0,$$

$$\Lambda_{+--+})\quad \partial_z^2\widehat{P}_n+\tfrac{3z^2-4}{z(z+1)(z-1)}\partial_z\widehat{P}_n+\tfrac{n(n+2)}{(z+1)(1-z)}\widehat{P}_n=0,$$

$$\Lambda_{+---})\quad \partial_z^2\widehat{P}_n+\tfrac{3z^2-2}{z(z+1)(1-z)}\partial_z\widehat{P}_n+\tfrac{n(4-n)}{(z+1)(z-1)}P_n=0.$$

The polynomial P_n of degree n exists for $\lambda_n\in\Lambda_b$ with n even, that is, $\Lambda_n=\{n\in\mathbb{Z}:16n^2+16n+3\}$, for $\Lambda_{++-+})$ and $\Lambda_{+--+})$. Therefore $E=E_n=\{n\in\mathbb{Z}:-4n^2-4n\}$.
The corresponding solutions for Λ_n are

$$\Lambda_{+++-})\qquad \Phi_{1,n}=\widehat{P}_{2n}\widehat{f}_n\Phi_{1,0},\quad \Phi_{1,0}=\frac{\sqrt[4]{(z^2-1)^5}}{z}\qquad \widehat{f}_n=1,\quad \widehat{\Psi}_{1,0}=z-\tfrac{1}{z},$$

$$\Lambda_{+--+})\qquad \widehat{\Phi}_{2,n}=\widehat{Q}_{2n}\widehat{f}_n\widehat{\Phi}_{2,0},\quad \widehat{\Phi}_{2,0}=\tfrac{z^2}{\sqrt[4]{z^2-1}}\qquad \widehat{f}_n=1\quad \widehat{\Psi}_{2,0}=\tfrac{z^2}{\sqrt{z^2-1}}.$$

These two solutions are equivalent to the same solution of the original Schrödinger equation and corresponds to the well known supersymmetric quantum mechanics approach to this Pöschl-Teller potential, [24, 25]. Furthermore, for all $\lambda\in\Lambda_n$,

$$\mathrm{DGal}(\widehat{\mathbf{L}}_\lambda/\mathbb{C}(x))=\mathbb{G}^{[4]},\quad \mathrm{DGal}(\widehat{L}_\lambda/\widehat{K})=\mathrm{DGal}(L_\lambda/K)=e,$$

$$\dim_{\mathbb{C}}\mathcal{E}(\widehat{\mathbf{H}}-\lambda)=2,\quad \dim_{\mathbb{C}}\mathcal{E}(\widehat{H}-\lambda)=\dim_{\mathbb{C}}\mathcal{E}(H-\lambda)=4.$$

Searching Potentials From Parameterized Differential Equations.

The main object to search new potentials using $\widehat{\partial}_z$ is the family of differential equations presented by Darboux in [28], see section 1.2.2 and equation (1.25), which can be written in the form

$$\partial_z^2\widehat{y}+\widehat{P}\partial_z\widehat{y}+(\widehat{Q}-\lambda\widehat{R})\widehat{y}=0,\quad \widehat{P},\widehat{Q},\widehat{R}\in\widehat{K}. \tag{2.13}$$

We recall that some Riemann's differential equations, presented in section 1.1.4, corresponds to this kind.

When we have a differential equation in the form (2.13), we reduce it to put it in the form of the reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\Phi = \lambda\Phi$, checking that $\mathrm{Card}(\Lambda) > 1$. Thus, starting with the potential $\widehat{\mathbf{V}}$ and arriving to the potential V we obtain the Schrödinger equation $H\Psi = \lambda\Psi$. This methodology (heuristic) is detailed below.

1. Reduce a differential equation of the form (2.13) and put it in the form $\widehat{\mathbf{H}}\Phi = \lambda\Phi$, checking that $\mathrm{Card}(\Lambda) > 1$ and to avoid triviality, α must be a non-constant function.
2. Write $\mathcal{W} = \frac{1}{4}\partial_z(\ln\alpha)$ and obtain $\widehat{V}(z) = \alpha(\widehat{\mathbf{V}} - \partial_z\mathcal{W} - \mathcal{W}^2)$.
3. Solve the differential equation $(\partial_x z)^2 = \alpha$, write $z = z(x)$, $V(x) = \widehat{V}(z(x))$.

To illustrate this method, we present the following examples.

Bessel Potentials

- (From Darboux transformations over $V = 0$) In the differential equation

$$\partial_z^2\Phi = \left(\frac{n(n+1)}{z^2} + \mu\right)\Phi, \quad \mu \in \mathbb{C},$$

  we see that $\lambda = -n(n+1)$ and $\alpha = z^2$. Applying the method, we obtain $\widehat{\mathbf{V}} = \mu$ we obtain $\widehat{V}(z) = \mu z^2 + \frac{1}{4}$ and $z = z(x) = e^{\pm x}$. Thus, we have obtained the potentials $V(x) = \widehat{V}(z(x)) = \mu e^{\pm 2x} + \frac{1}{4}$ (compare with [36, §6.9]).

- (From Bessel differential equation) The equation

$$\partial_z^2 y + \frac{1}{z}\partial_z y + \frac{z^2 - n^2}{z^2} y = 0, \quad n \in \frac{1}{2} + \mathbb{Z},$$

  is transformed to the reduced equation

$$\partial_z^2\Phi = \left(\frac{n^2}{z^2} - \frac{4z^2+1}{4z^2}\right)\Psi.$$

  We can see that $\lambda = -n^2$, $\alpha = z^2$, obtaining $\widehat{\mathbf{V}} = -z^2 - \frac{1}{4}$, $\widehat{V} = -z^4 - \frac{1}{4}z^2 + \frac{1}{4}$ and $z = z(x) = e^{\pm x}$. Thus, we have obtained the potential $V(x) = \widehat{V}(z(x)) = -e^{\pm 4x} - \frac{1}{4}e^{\pm 2x} + \frac{1}{4}$ (compare with [36, §6.9]).

We remark that the previous examples give us potentials related with the Morse potential, due to their solutions are given in term of Bessel functions.

We can apply this method to equations such as Whittaker, Hypergeometric and in particular, differential equations involving orthogonal polynomials (compare with [24, §5]).

Final Remark

The aim of this work is to give, in contemporary terms, a formalization of original ideas and intuitions given by G. Darboux, E. Witten and L. É. Gendenshteïn in the context of the Galois theory of linear differential equations. We found the following facts.

- The superpotential is an algebraic solution of the Riccati equation associated with a potential, defined over a differential field.
- Darboux transformation was interpreted as an isogaloisian transformation, allowing to obtain isomorphisms between their eigenrings.
- We introduced in a general way the Hamiltonian algebrization method,which in particular allow to apply algorithmic tools such as Kovacic's algorithm to obtain the solutions, differential Galois groups and Eigenrings of second order linear differential equations. We applied successfully this algebrization procedure to solve problems in Supersymmetric quantum mechanics.
- We can construct algebraically solvable and non-trivial algebraically quasi-solvable potentials in the following ways.
  1. Giving the potential in where for $\lambda = \lambda_0$ the Schrödinger equation is integrable. After we put $\lambda \neq \lambda_0$ checking that the Schrödinger equation is integrable for more than one value of the parameter λ.
  2. Giving a superpotential to obtain the potential and after we check if the Schrödinger equation is integrable for more than one value of the parameter λ.
  3. Since parameterized second order linear differential equations applying an inverse process in the Hamiltonian algebrization method. In particular, we can use algebraically solvable and algebraically quasi-solvable potentials, special functions with parameters (in particular with polynomial solutions).

This thesis is a starting point to analyze quantum theories through Galoisian theories. Therefore open questions and future work arise in a natural way:

supersymmetric quantum mechanics with dimension greater than 2, relationship between algebraic and analytic spectrums, etc.

As a conclusion, as happen in other areas of the field of differential equations, in view of the many families of examples studied along this thesis, we can conclude that the *differential Galois theory* is a natural framework in which some aspects of *supersymmetric quantum mechanics* may appear more clearly.

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